HOMEOMORPHISMS OF CONTINUA THROUGH PROJECTIVE FRAÏSSÉ LIMITS

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ABSTRACT. We study homeomorphisms and the homeomorphism groups of compact metric spaces using the automorphism groups of projective Fraïssé limits. In our applications, we investigate the Polish group $\mathrm{Homeo}(P)$ of all homeomorphisms of the pseudoarc P using the automorphism group $\mathrm{Aut}(\mathbb{P})$ of the pre-pseudoarc $\mathbb{P}$. Strengthening results from the literature, we show that the diagonal conjugacy action of $\mathrm{Homeo}(P)$ on $\mathrm{Homeo}(P)^{\mathbb{N}}$ has a dense orbit. In our second application, we show that there exists a homeomorphism of P that is not conjugate in $\mathrm{Homeo}(P)$ to an element of $\mathrm{Aut}(\mathbb{P})$.

§ 1. INTRODUCTION

As an application of our methods based on projective Fraïssé theory, we prove the following theorem that is phrased purely in terms of the homeomorphism group of the pseudoarc. (We provide some information on the pseudoarc in the introduction. A stronger version of the theorem below is proved as Theorem 4.1.)

Theorem 1.1. *For each natural number $n \geq 1$, the diagonal conjugacy action of* $\mathrm{Homeo}(P)$ *on* $\mathrm{Homeo}(P)^n$ *has a dense orbit. In fact, the diagonal conjugacy action of* $\mathrm{Homeo}(P)$ *on* $\mathrm{Homeo}(P)^{\mathbb{N}}$ *has a dense orbit.*

The statement above was conjectured for $n = 1$ by Kwiatkowska [11] and was established, again in the case $n = 1$, by Bice and Malicki [2]. We prove the general statement for arbitrary n, in fact, for $\mathbb{N}$, using an argument based on projective Fraïssé theory. Our methods are different from those of [2]. They also appear to be simpler.

Theorem 1.1 contributes to the study of the existence of dense orbits in the diagonal conjugacy actions of Polish groups G on their finite G^n, $n \in \mathbb{N}$, $n \geq 1$, and infinite $G^{\mathbb{N}}$ products. Such actions are defined by the formula

$$G \times G^I \ni (g, (h_i)_{i\in I}) \mapsto (gh_ig^{-1})_{i\in I},$$

where $I = n \geq 1$ or $I = \mathbb{N}$. This line of investigation was initiated by Glasner and Weiss [6]; groups whose diagonal conjugacy actions with $n = 1$ had a dense orbit were said to possess the topological Rohlin property in that paper. A general theory behind the existence of dense orbits for diagonal conjugacy actions for automorphism groups of countable structures was developed by Kechris and Rosendal in [8]. A survey of the area can be found in [5]. The reader may consult the papers

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[1], [2], [4], [5], [6], [8], [10], [11], [12], [15] for examples of Polish groups with dense orbits in diagonal conjugacy actions.

However, the goal of this paper is broader than merely proving Theorem 1.1—we investigate the relationship between the automorphism groups of projective Fraïssé limits and the homeomorphism groups of compact spaces that are canonical quotients of those limits. The mathematical context for this investigation is as follows. One starts with a countable (up to isomorphism) family $\mathcal{K}$ of finite reflexive graphs and a family of epimorphisms among graphs in $\mathcal{K}$. When writing $\mathcal{K}$ we have in mind the family of reflexive graphs together with the family of epimorphisms. The binary reflexive graph relation on the structures in $\mathcal{K}$ is denoted by R. Assuming that the epimorphisms in $\mathcal{K}$ fulfill a projective amalgamation condition, a canonical projective limit $\mathbb{K}$ of $\mathcal{K}$ exists. The object $\mathbb{K}$ is a compact totally disconnected metric space that can be equipped with a canonical interpretation $R^{\mathbb{K}}$ of R. This is a compact binary reflexive graph relation on $\mathbb{K}$, and with a set of continuous epimorphisms from $\mathbb{K}$ to the structures in $\mathcal{K}$. If $R^{\mathbb{K}}$ is transitive, then it is a compact equivalence relation, which allows us to form the quotient space $K = \mathbb{K}/R^{\mathbb{K}}$, which is a compact metric space. Even though $\mathbb{K}$ as a topological space is totally disconnected, the quotient space K is often connected, that is, it is a continuum; in fact, this is the most interesting situation from the topologically point of view.

We consider two groups associated with the objects above, the automorphism group $\mathrm{Aut}(\mathbb{K})$ of $\mathbb{K}$ and the homeomorphism group $\mathrm{Homeo}(K)$ of the quotient space K. Both these groups come with natural Polish group topologies on them. Furthermore, there is a natural continuous homomorphism

$$\mathrm{pr}\colon \mathrm{Aut}(\mathbb{K}) \to \mathrm{Homeo}(K).$$

Elements of $\mathrm{Aut}(\mathbb{K})$ are easier to deal with than elements of $\mathrm{Homeo}(K)$ as they are essentially combinatorial objects. Likewise, the group $\mathrm{Aut}(\mathbb{K})$ is easier to deal with than the group $\mathrm{Homeo}(K)$, for example, it is non-archimedean. We exploit this drop in complexity when passing from $\mathrm{Homeo}(K)$ to $\mathrm{Aut}(\mathbb{K})$ in order to study $\mathrm{Homeo}(K)$ through $\mathrm{Aut}(\mathbb{K})$. For more precise information on projective Fraïssé theory, see Appendix A.

To achieve this goal, we need to consider the topology on $\mathrm{Aut}(\mathbb{K})$ that is obtained by pulling back the topology on $\mathrm{Homeo}(K)$ using the continuous homomorphism pr. By continuity of pr, this topology is weaker than the natural topology on $\mathrm{Aut}(\mathbb{K})$. In Section 2, we give a combinatorial description of the topology on $\mathrm{Aut}(\mathbb{K})$ inherited from $\mathrm{Homeo}(K)$ through pr.

Then we move to finding combinatorial conditions corresponding to the existence of dense orbits under diagonal conjugacy actions. Since we will be assuming that $\mathrm{Aut}(\mathbb{K})$ is dense in $\mathrm{Homeo}(K)$, when considering density of diagonal conjugacy actions, it suffices to consider conjugacy by elements of $\mathrm{Aut}(\mathbb{K})$. This leads to three types of conjugacy actions. Given $n \in \mathbb{N}$, $n \geq 1$, we consider:

(a) conjugacy by elements of $\mathrm{Aut}(\mathbb{K})$ of tuples in $\mathrm{Aut}(\mathbb{K})^n$;
(b) conjugacy by elements of $\mathrm{Aut}(\mathbb{K})$ of tuples in $\mathrm{Aut}(\mathbb{K})^n$ taken with the topology inherited from $\mathrm{Homeo}(K)^n$;
(c) conjugacy by elements of $\mathrm{Aut}(\mathbb{K})$ of tuples in $\mathrm{Homeo}(K)^n$.

Observe that having a dense orbit with respect to an action of type (a) implies the existence of a dense orbit in the corresponding action of type (b), which, in turn, implies the existence of a dense orbit with respect to the action of type (c).

We obtain combinatorial conditions equivalent to the following properties of the actions as in (a)–(c):

(a′) there exist comeagerly many $\overline{\gamma} \in \mathrm{Aut}(\mathbb{K})^n$ with conjugacy orbits dense in $\mathrm{Aut}(\mathbb{K})^n$;
(b′) there exist comeagerly many $\overline{\gamma} \in \mathrm{Aut}(\mathbb{K})^n$ with conjugacy orbits dense in $\mathrm{Homeo}(K)^n$;
(c′) there exist comeagerly many $\overline{\gamma} \in \mathrm{Homeo}(K)^n$ with conjugacy orbits dense in $\mathrm{Homeo}(K)^n$.

The three combinatorial conditions are related but distinct. They are versions of the Joint Projection Property for suitable categories and are obtained by dualizing to the projective setting and adapting to the mix of two topologies (coming from $\mathrm{Aut}(\mathbb{K})$ and $\mathrm{Homeo}(K)$) of the Joint Embedding Property in the paper by Kechris and Rosendal [8]. Our conditions are stated, and the theorems establishing the equivalence between the appropriate Joint Projection Property and the existence of dense orbits in the corresponding conjugacy action are proved in Section 3. We note that case (b)/(b′) exhibits the most interesting interaction between the two topologies on $\mathrm{Aut}(\mathbb{K})$: comeagerness refers to the natural topology on $\mathrm{Aut}(\mathbb{K})$, while density of orbits concerns the topology inherited from $\mathrm{Homeo}(K)$. This is precisely the case applied to the pseudoarc. Finally, note that since the property of a point having a dense orbit under a continuous action of a Polish group is G_δ, the properties in (a′) and (c′) are equivalent to the existence of a single dense orbit in $\mathrm{Aut}(\mathbb{K})^n$ and $\mathrm{Homeo}(K)^n$, respectively.

For our applications, recall that the pseudoarc is a continuum, that is, a compact connected metric space, constructed by Knaster [9] and characterized by Bing [3] as the unique chainable hereditarily indecomposable continuum. In the same paper [3], Bing gave another characterization of the pseudoarc. It is the unique continuum that is generic in the following sense. Let $\mathcal{C}$ be the space of all continua included in the Hilbert cube $[0,1]^{\mathbb{N}}$ endowed with the Vietoris topology, which is induced by the Hausdorff metric. The space $\mathcal{C}$ is a compact metric space. It turns out [3] that there exists a continuum P such that the subset of $\mathcal{C}$ consisting of copies of P is comeager in $\mathcal{C}$, in fact, it is a dense G_δ. This continuum is the pseudoarc. For more information on the pseudoarc, the reader may consult the survey [13] or the book [14].

Consider the family $\mathcal{P}$ consisting of all structures isomorphic to reflexive graphs of the following form:

$$L = \{0, \dots, n\} \text{ with } xR^L y \Leftrightarrow |x - y| \leq 1, \text{ for } x, y \in L,$$

where $n \in \mathbb{N}$. Morphisms in $\mathcal{P}$ are all epimorphisms among structures in $\mathcal{P}$. As proved in [7], the family $\mathcal{P}$ forms a transitive projective Fraïssé class and its limit $\mathbb{P}$ is such that $P = \mathbb{P}/R^{\mathbb{P}}$ is homeomorphic to the pseudoarc.

In Section 4, we apply the results of Section 3 to the pseudoarc. We prove the appropriate Joint Projection Property for the class $\mathcal{P}$. This allows us to apply the results of Section 3 to show that there there exists an element of $\mathrm{Aut}(\mathbb{P})^{\mathbb{N}}$ whose orbit with respect to the diagonal conjugacy action $\mathrm{Aut}(\mathbb{P})$ on $\mathrm{Aut}(\mathbb{P})^{\mathbb{N}}$ is dense in $\mathrm{Homeo}(P)^{\mathbb{N}}$; in fact, the set of such tuples is comeager in $\mathrm{Aut}(\mathbb{P})^{\mathbb{N}}$. In particular, Theorem 1.1 follows.

Finally, in Section 5, we turn to the following natural problem concerning the conjugacy action of $\mathrm{Homeo}(P)$ on itself: is every homeomorphism in $\mathrm{Homeo}(P)$ conjugate in $\mathrm{Homeo}(P)$ to an element of $\mathrm{Aut}(\mathbb{P})$? We answer it in the negative by exhibiting a homeomorphism whose conjugacy class misses $\mathrm{Aut}(\mathbb{P})$. This is the first example of this phenomenon in the projective Fraïssé context. The construction uses our analysis of topologies in Section 2. The homeomorphism is constructed by finding a sequence of automorphisms in $\mathrm{Aut}(\mathbb{P})$ that is Cauchy with respect to the uniformity inducing the topology on $\mathrm{Aut}(\mathbb{P})$ that is inherited from $\mathrm{Homeo}(P)$ through the map pr.

Notation and conventions. A short exposition of the projective Fraïssé theory is given in Appendix A. With this in mind and following Appendix A, we fix notation and some conventions for the paper.

- — $\mathcal{K}$ is a countable projective Fraïssé family;
- — $\mathbb{K}$ with the binary relation $R^{\mathbb{K}}$ is the projective Fraïsse limit of $\mathcal{K}$;
- — if $R^{\mathbb{K}}$ is transitive, then $K = \mathbb{K}/R^{\mathbb{K}}$ is the quotient topological space and $\mathrm{pr} : \mathbb{K} \to K$ is the canonical projection.

By $\mathrm{Aut}(\mathbb{K})$ we denote the automorphism group induced by $\mathcal{K}$ and again by pr we denote the continuous homomorphism $\mathrm{pr}\colon \mathrm{Aut}(\mathbb{K}) \to \mathrm{Homeo}(K)$ induced by the quotient map $\mathrm{pr} : \mathbb{K} \to K$.

We will often identify $\mathrm{Aut}(\mathbb{K})$ with its image under the homomorphism pr in $\mathrm{Homeo}(K)$, that is, we will consider $f \in \mathrm{Aut}(\mathbb{K})$ as an element of $\mathrm{Homeo}(K)$ keeping in mind the identification spelled out in (A.1) in Appendix A. Note that this constitutes some abuse of terminology since, in general, pr need not be injective.

§ 2. Topologies on $\mathrm{Aut}(\mathbb{K})$

We start with a lemma that will be useful in several places.

Lemma 2.1. *Assume that $\mathcal{K}$ is transitive. Fix $k \in \mathbb{N}$, $k \geq 1$. For each epimorphism $\phi\colon \mathbb{K} \to A$, there exist epimorphisms $\psi\colon \mathbb{K} \to B$ and $f\colon B \to A$ such that*

(i) $\phi = f \circ \psi$;

(ii) if $a, b \in B$ and $aR^k b$, then $f(a)Rf(b)$.

Proof. We can assume that there exists a generic sequence $\pi_{i,i+1}\colon A_{i+1} \to A_i$ with $\mathbb{K} = \operatorname{proj\,lim}_i(A_i, \pi_{i,i+1})$ and $\phi = \pi_0\colon \mathbb{K} \to A_0 = A$. Assume we have a sequence (j_l) of natural numbers with $0 < j_l < j_{l+1}$ and points $a_l, b_l \in A_{j_l}$ with

$$a_l R^k b_l \ \text{ and } \ \neg\big(\pi_{0,j_l}(a_l)\, R\, \pi_{0,j_l}(b_l)\big). \tag{2.1}$$

By going to subsequences, we can assume that for some $a, b \in A_0$ and all l,

$$\pi_{j_l,j_{l+1}}(a_{l+1}) = a_l,\ \pi_{j_l,j_{l+1}}(b_{l+1}) = b_l,\ \pi_{0,j_l}(a_l) = a,\ \text{and}\ \pi_{0,j_l}(b_l) = b. \tag{2.2}$$

We now fix sequences $x_l, y_l \in \mathbb{K}$, $l \in \mathbb{N}$, with $\pi_{j_l}(x_l) = a_l$ and $\pi_{j_l}(y_l) = b_l$, and, by compactness, assume that they converge to x and y, respectively. By (2.1) and (2.2), we have that $\pi_{j_l}(x)R^k\pi_{j_l}(y)$, for all l, so xR^ky. Since R is transitive on $\mathbb{K}$, we get xRy. This condition implies, by (2.2), that aRb, which contradicts (2.1), and the lemma follows. $\square_{\text{Lemma2.1}}$

Fix a metric d^0 on K. The metric induces the supremum metric

$$d(f,g) = \sup\{d^0(f(x), g(x)) :\ x \in K\} \quad (f, g \in \mathcal{C}(K,K)).$$

By the same letter d, we denote the pseudometric on $\mathrm{Aut}(\mathbb{K})$ given by

$$d(f,g) = d\big(\mathrm{pr}(f), \mathrm{pr}(g)\big).$$

Let

$$\mathrm{Epi}_{\mathbb{K}} = \bigcup_{A\in\mathcal{K}} \mathrm{Epi}(\mathbb{K}, A).$$

With each epimorphism $\phi \in \mathrm{Epi}_{\mathbb{K}}$, we associate the set

$$U_\phi = \{(f,g) \in \mathrm{Aut}(\mathbb{K}) \times \mathrm{Aut}(\mathbb{K}) \mid \phi\circ f = \phi \circ g\}$$

and for $k \in \mathbb{N}$, $k > 0$, the set

$$U_\phi^{(k)} = \{(f,g) \in \mathrm{Aut}(\mathbb{K}) \times \mathrm{Aut}(\mathbb{K}) \mid \phi\circ f\, R^k\, \phi \circ g\}$$

We will write $U_\phi^{(0)}$ for U_ϕ. For $k \in \mathbb{N}$, let

$$\mathcal{U}^{(k)}$$

consist of all subsets of $\mathrm{Aut}(\mathbb{K}) \times \mathrm{Aut}(\mathbb{K})$ containing a set of the form $U_\phi^{(k)}$.

Lemma 2.2. *Fix $k \in \mathbb{N}$, $k \geq 1$.*

(i) $U_\phi^{(1)} \subseteq U_\phi^{(k)}$*, for each* $\phi \in \mathrm{Epi}_{\mathbb{K}}$.
(ii) For each $\phi \in \mathrm{Epi}_{\mathbb{K}}$*, there exists* $\psi \in \mathrm{Epi}_{\mathbb{K}}$ *such that*

$$U_\psi^{(k)} \subseteq U_\phi^{(1)}.$$

Proof. Point (i) is immediate from the definition of $U_\phi^{(k)}$. Point (ii) is a consequence of Lemma 2.1. $\square$

Recall that a **uniformity** on a set X is a family $\mathcal{U}$ of subsets of $X \times X$ such that

— $\{(x,x) \mid x \in X\} \subseteq U$ for each $U \in \mathcal{U}$;
— for each $V \in \mathcal{U}$, there exists $U \in \mathcal{U}$, such that $U \circ U \subseteq V$;
— if $U \in \mathcal{U}$, then $U^{-1} \in \mathcal{U}$;
— if $U \in \mathcal{U}$ and $U \subseteq V$, then $V \in \mathcal{U}$.

If ρ is a pseudometric on a set X, the **uniformity induced by** ρ is the family of all subsets of $X \times X$ containing sets of the form

$$\{(x,y) \in X \times X \mid \rho(x,y) < r\}, \text{ for } r > 0.$$

Theorem 2.3. *Assume $\mathcal{K}$ is transitive.*

(i) $\mathcal{U}^{(k)}$ *is a uniformity on* $\mathrm{Aut}(\mathbb{K})$*, for each* $k \in \mathbb{N}$.
(ii) The uniformity $\mathcal{U}^{(0)}$ *is equal to the uniformity induced by the uniform metric on* $\mathrm{Aut}(\mathbb{K})$.
(iii) For each $k \geq 1$*, the uniformity* $\mathcal{U}^{(k)}$ *is equal to the uniformity induced by the pseudometric* d *on* $\mathrm{Aut}(\mathbb{K})$.

Proof. We only handle the case $k > 0$, which is somewhat trickier than $k = 0$. To see point (i) for $k > 0$ and point (iii), it suffices to show that

$$\forall \epsilon > 0\ \exists \phi \in \mathrm{Epi}_{\mathbb{K}}\ \ U^{(k)}_{\phi} \subseteq \{(\sigma, \tau) \in \mathrm{Aut}(\mathbb{K}) \times \mathrm{Aut}(\mathbb{K}) \mid d(\sigma, \tau) < \epsilon\} \tag{2.3}$$

and, conversely,

$$\forall \phi \in \mathrm{Epi}_{\mathbb{K}}\ \exists \epsilon > 0\ \ \{(\sigma, \tau) \in \mathrm{Aut}(\mathbb{K}) \times \mathrm{Aut}(\mathbb{K}) \mid d(\sigma, \tau) < \epsilon\} \subseteq U^{(k)}_{\phi}. \tag{2.4}$$

By Lemma 2.2, it suffices to show the statements above for $k = 1$ only.

We show (2.3) for $k = 1$ first. Fix $\epsilon > 0$. We prove that there exists $\phi \in \mathrm{Epi}_{\mathbb{K}}$ such that

$$\forall x, y \in \mathbb{K}\ \big(\phi(x) R \phi(y) \Rightarrow d^0(\pi(x), \pi(y)) < \epsilon/2\big). \tag{2.5}$$

Note that this statement implies that, for all $\sigma, \tau \in \mathrm{Aut}(\mathbb{K})$,

$$\phi \circ \sigma\, R\, \phi \circ \tau \Rightarrow d(\sigma, \tau) \leq \epsilon/2 < \epsilon,$$

and (2.3) for $k = 1$ is proved.

We proceed to proving (2.5). Since pr is continuous and $\mathbb{K}$ is compact, there exists $\delta > 0$ such that for $x, y \in \mathbb{K}$, if $d(x, y) < \delta$, then $d^0(\mathrm{pr}(x), \mathrm{pr}(y)) < \epsilon/2$. Let now $\phi\colon \mathbb{K} \to A$ be an epimorphism such that preimages of points have diameter $< \delta$. We claim that this ϕ works. Let $x, y \in \mathbb{K}$ be such that $\phi(x) R \phi(y)$. Since ϕ is an epimorphism, there exist $x', y' \in \mathbb{K}$ such that

$$\phi(x') = \phi(x),\ \phi(y') = \phi(y),\ \text{and } x' R y'.$$

Then, by our choice of ϕ, $d(x, x') < \delta$ and $d(y, y') < \delta$, so $d^0(\mathrm{pr}(x), \mathrm{pr}(x')) < \epsilon/2$ and $d^0(\mathrm{pr}(y), \mathrm{pr}(y')) < \epsilon/2$. Since $x' R y'$, we have $\mathrm{pr}(x') = \mathrm{pr}(y')$. It follows that $d^0(\mathrm{pr}(x), \mathrm{pr}(y)) < 2\epsilon/2 = \epsilon$.

Now, we show (2.4) for $k = 1$. Fix an epimorphism $\phi\colon \mathbb{K} \to A$. We are looking for $\epsilon > 0$ to satisfy the inclusion in (2.4). Consider $a, b \in A$ such that $\neg(aRb)$. Note that $\phi^{-1}(a)$ and $\phi^{-1}(b)$ are clopen subsets of $\mathbb{K}$ with

$$R\big(\phi^{-1}(a)\big) \cap \phi^{-1}(b) = \emptyset.$$

It follows that $\mathrm{pr}\big(\phi^{-1}(a)\big)$ and $\mathrm{pr}\big(\phi^{-1}(b)\big)$ are disjoint compact subsets of K, so they are at a positive d^0 distance $\epsilon_{a,b} > 0$ from each other. Since A is finite, we can let $\epsilon > 0$ be the minimum of all $\epsilon_{a,b}$ for $a, b \in A$ with $\neg(aRb)$. Now, if $\sigma, \tau \in \mathrm{Aut}(\mathbb{K})$ are such that $d(\sigma, \tau) < \epsilon$, then, for each $x \in \mathbb{K}$, $d^0\big(\mathrm{pr}(\sigma(x)), \mathrm{pr}(\tau(x))\big) < \epsilon$, so, by our choice of ϵ, we have $\sigma(x) R \tau(x)$, for each $x \in \mathbb{K}$; thus, $(\sigma, \tau) \in U^{(1)}_{\phi}$, as required. □

We state two immediate corollaries of Theorem 2.3.

Corollary 2.4. *Assume $\mathcal{K}$ is transitive. Let (σ_n) be a sequence in* $\mathrm{Aut}(\mathbb{K})$*.*

(i) If (σ_n) is d-Cauchy in $\mathcal{C}(K, K)$, then for each $\varphi \in \mathrm{Epi}(\mathbb{K}, A)$, there exists N such that

$$\forall n, m \geq N\ \forall x \in \mathbb{K}\ \big(\varphi(\sigma_n(x)) R \varphi(\sigma_m(y))\big).$$

(ii) If for each $\varphi \in \mathrm{Epi}(\mathbb{K}, A)$, there exists N such that

$$\forall n, m \geq N\ \forall x \in \mathbb{K}\ \big(\varphi(\sigma_n(x)) R \varphi(\sigma_m(x))\big),$$

then the sequence (σ_n) is d-Cauchy in $\mathcal{C}(K, K)$.

Proof. The conclusions are immediate from Theorem 2.3 (ii) and (iii). □

With each pair of epimorphisms $\phi, \psi \in \mathrm{Epi}_{\mathbb{K}}$, we associate the following subsets of $\mathrm{Aut}(\mathbb{K})$:

$$B_{\psi,\phi} = \{\tau \in \mathrm{Aut}(\mathbb{K}) : \ \psi = \phi \circ \tau\},$$

and for $k \in \mathbb{N}$, $k \geq 1$,

$$B^{(k)}_{\psi,\phi} = \{\tau \in \mathrm{Aut}(\mathbb{K}) : \psi \ R^k \ \phi \circ \tau\}.$$

We may also write $B^{(0)}_{\psi,\phi}$ instead of $B_{\psi,\phi}$.

Recall that a family $\mathcal{B}$ of subsets of a topological space X is called a **neighborhood basis** if for each $x \in X$ and open set $U \subseteq X$ with $x \in U$ there exists $B \in \mathcal{B}$ such that $B \subseteq U$ and x is in the interior of B.

Corollary 2.5.

(i) Sets $B_{\phi,\psi}$, with $\phi, \psi \in \mathrm{Epi}_{\mathbb{K}}$, form a clopen neighborhood basis of the topology on $\mathrm{Aut}(\mathbb{K})$.

(ii) Fix $d \geq 1$. Sets $B^{(d)}_{\phi,\psi}$, with $\phi, \psi \in \mathrm{Epi}_{\mathbb{K}}$, form a neighborhood basis of the topology on $\mathrm{Aut}(\mathbb{K})$ *induced by the pseudometric d, that is, the topology inherited from* $\mathrm{Homeo}(K)$.

Proof. Point (i) follows from Theorem 2.3 (ii) and point (ii) from Theorem 2.3 (iii). We give details for the latter argument. By Theorem 2.3 (iii), for $\sigma \in \mathrm{Aut}(\mathbb{K})$, sets of the form

$$\{\tau \in \mathrm{Aut}(\mathbb{K}) \mid (\phi \circ \sigma) \, R^d \, (\phi \circ \tau)\}, \text{ for } \phi \in \mathrm{Epi}_{\mathbb{K}},$$

are a neighborhood basis at σ of the topology induced by d containing σ in their interiors. Setting $\psi = \phi \circ \sigma$, the conclusion follows. $\square_{\text{Corollary2.5}}$

§ 3. Joint Projection Property

Let $\mathcal{K}$ be a category. Following Kechris–Rosendal [8, Section 2], and adapting their work to the projective setting, we define the category $\mathcal{K}_{\mathrm{p}}$ as follows.

Definition 3.1.

Objects: $(A, B, f, g) \in \mathcal{K}_{\mathrm{p}}$ iff $A, B \in \mathcal{K}$, $f, g \in \mathrm{Epi}(A, B)$.
Morphisms: The $\mathcal{K}$-epimorphism $\alpha : A \to A'$ is an epimorphism in $\mathcal{K}_{\mathrm{p}}$ between (A, B, f, g) and (A', B', f', g') iff

$$\forall a_0, a_1 \in A \ \big(f(a_0) = g(a_1) \ \Rightarrow \ f'(\alpha(a_0)) = g'(\alpha(a_1))\big).$$

It is not difficult to see that α is an epimorphism in $\mathcal{K}_{\mathrm{p}}$ between (A, B, f, g) and (A', B', f', g') precisely when there exists an $\mathcal{K}$-epimorphism $\beta : B \to B'$, such that

$$\beta \circ f = f' \circ \alpha \ \text{ and } \ \beta \circ g = g' \circ \alpha.$$

We now modify the definition of morphism in $\mathcal{K}_p$ to obtained the definition of an approximate morphism in this category.

Definition 3.2. Let (A, B, f, g) and $(A', B', f', g') \in \mathcal{K}_{\mathrm{p}}$. The $\mathcal{K}$-epimorphism α between A and A' is an *approximate* epimorphism in $\mathcal{K}_{\mathrm{p}}$ between (A, B, f, g) and (A', B', f', g') iff

$$\forall a_0, a_1 \in A : \ f(a_0) = g(a_1) \ \Rightarrow \ f'(\alpha(a_0)) \ R \ g'(\alpha(a_1)).$$

Definition 3.3. For $n \in \mathbb{N}$, we let $\mathcal{K}_{\mathrm{p}}^{\times n}$ denote the class of objects of the form $(A, B, f_0, \ldots, f_{n-1}, g_0, \ldots, g_{n-1})$, where $(A, B, f_i, g_i) \in \mathcal{K}_{\mathrm{p}}$, for each $i < n$.

We call $\alpha \in \mathrm{Epi}(A, A')$ an epimorphism (an approximate epimorphism, respectively) between

$$(A, B, f_0, \ldots, f_{n-1}, g_0, \ldots, g_{n-1})$$

and

$$(A', B', f'_0, \ldots, f'_{n-1}, g'_0, \ldots, g'_{n-1})$$

if for each i the map α is an epimorphism (an approximate epimorphism, respectively) between (A, B, f_i, g_i) and (A', B', f'_i, g'_i).

When n is clear from the context, we write

$$(A, B, \overline{f}, \overline{g}) \text{ for } (A, B, f_0, \ldots, f_{n-1}, g_0, \ldots, g_{n-1}).$$

Fix n. We say that $\mathcal{K}_{\mathrm{p}}^{\times n}$ has

(a) JPP,
(b) half-approximate JPP,
(c) approximate JPP,

if for all $(A, B, \overline{f}, \overline{g})$ and $(A', B', \overline{f'}, \overline{g'})$ in $\mathcal{K}_{\mathrm{p}}^{\times n}$, there exist $(A^+, B^+, \overline{f^+}, \overline{g^+})$ in $\mathcal{K}_{\mathrm{p}}^{\times n}$ and

$$\alpha\colon (A^+, B^+, \overline{f^+}, \overline{g^+}) \to (A, B, \overline{f}, \overline{g}) \text{ and } \alpha'\colon (A^+, B^+, \overline{f^+}, \overline{g^+}) \to (A', B', \overline{f'}, \overline{g'})$$

such that

(a) α and α' are epimorphisms,
(b) α is an epimorphism and α' is an approximate epimorphism,
(c) α and α' are approximate epimorphisms.

We now come to the main theorem of this section. For a sequence $\overline{\gamma} = (\gamma_j)_{j<n}$ of elements of a group G, we write

$$\overline{\gamma}^G = \{(g\gamma_j g^{-1})_{j<n} \mid g \in G\} \subseteq G^n.$$

Theorem 3.4. *Suppose that $\mathcal{K}$ is a transitive projective Fraïssé class with the property that* $\mathrm{Aut}(\mathbb{K})$ *has a dense image under the canonical homomorphism into* $\mathrm{Homeo}(K)$*. Let $n \in \mathbb{N}$.*

(i) If $\mathcal{K}_p^{\times n}$ has the approximate JPP, then, for a non-empty open set $U \subseteq$ $\mathrm{Homeo}(K)^n$*,*

$$\{\overline{\gamma} \in \mathrm{Aut}(\mathbb{K})^n : \overline{\gamma}^{\mathrm{Aut}(\mathbb{K})} \cap U \neq \emptyset\} \text{ is dense in } \mathrm{Homeo}(K)^n.$$

(ii) If $\mathcal{K}_p^{\times n}$ has the half-approximate JPP, then, for a non-empty open set $U \subseteq$ $\mathrm{Homeo}(K)^n$*,*

$$\{\overline{\gamma} \in \mathrm{Aut}(\mathbb{K})^n : \overline{\gamma}^{\mathrm{Aut}(\mathbb{K})} \cap U \neq \emptyset\} \text{ is dense in } \mathrm{Aut}(\mathbb{K})^n.$$

(iii) If $\mathcal{K}_p^{\times n}$ has the JPP, then for a non-empty open set $U \subseteq$ $\mathrm{Aut}(\mathbb{K})^n$*,*

$$\{\overline{\gamma} \in \mathrm{Aut}(\mathbb{K})^n : \overline{\gamma}^{\mathrm{Aut}(\mathbb{K})} \cap U \neq \emptyset\} \text{ is dense in } \mathrm{Aut}(\mathbb{K})^n.$$

Note that *(iii)* is the dualized version of Kechris-Rosendal.

First we note that with epimorphisms $f, g\colon A \to B$ and $\varphi \in \mathrm{Epi}(\mathbb{K}, A)$ we can naturally associate the clopen set $B_{f\circ\varphi, g\circ\varphi}$, for which we introduce the following shorthand notation.

Definition 3.5. If $f, g\colon A \to B$ are epimorphisms and $\varphi \in \mathrm{Epi}(\mathbb{K}, A)$, then we let

— $B_{f,g;\varphi} = B_{f\circ\varphi, g\circ\varphi}$, that is, $\sigma \in B_{f,g;\varphi}$ iff $(g \circ \varphi) \circ \sigma = f \circ \varphi$,

— $B^{(k)}_{f,g;\varphi} = B^{(k)}_{f\circ\varphi, g\circ\varphi}$, that is, $\sigma \in B^{(k)}_{f,g;\varphi}$ iff $(g \circ \varphi) \circ \sigma \; R^k \; f \circ \varphi$.

Before embarking on proving the theorem, we need several lemmas, the first one of which describes the behavior, relevant to our proof, of sets of the form $B^{(k)}_{f,g;\varphi}$ under conjugation.

Lemma 3.6. *Let (A, B, f, g) and $(A', B', f', g') \in \mathcal{K}_{\mathrm{p}}$. Suppose that*

$$\alpha \in \mathrm{Epi}(A, A'), \;\; \varphi \in \mathrm{Epi}(\mathbb{K}, A), \;\; \varphi' \in \mathrm{Epi}(\mathbb{K}, A'), \;\; \textit{and} \;\; \sigma \in B_{\alpha\circ\varphi,\varphi'}.$$

If α is an approximate $\mathcal{K}_p$-epimorphism between (A, B, f, g) and (A', B', f', g'), then

$$\sigma B^{(0)}_{f,g;\varphi} \sigma^{-1} \subseteq B^{(1)}_{f',g';\varphi'} \tag{3.1}$$

If α is a $\mathcal{K}_p$-epimorphism, then

$$\sigma B^{(1)}_{f,g;\varphi} \sigma^{-1} \subseteq B^{(1)}_{f',g';\varphi'}, \tag{3.2}$$

and

$$\sigma B^{(0)}_{f,g;\varphi} \sigma^{-1} \subseteq B^{(0)}_{f',g';\varphi'}. \tag{3.3}$$

Proof. We prove (3.1) first. Given $\sigma^* \in B^{(0)}_{f\circ\varphi, g\circ\varphi}$, we need to check that

$$\sigma\sigma^*\sigma^{-1} \in B^{(1)}_{f',g';\varphi'}.$$

Since $\sigma^* \in B^{(0)}_{f,g;\varphi}$ means that $f \circ \varphi \;=\; g \circ \varphi \circ \sigma^*$, we have

$$(f' \circ \alpha \circ \varphi) \; R \; (g' \circ \alpha \circ \varphi \circ \sigma^*). \tag{3.4}$$

On the other hand, $\sigma \in B_{\alpha\circ\varphi,\varphi'}$ means that $\alpha \circ \varphi = \varphi' \circ \sigma$, which gives

$$f' \circ \alpha \circ \varphi = f' \circ \varphi' \circ \sigma \;\; \text{and} \;\; g' \circ \alpha \circ \varphi \circ \sigma^* = g' \circ \varphi' \circ \sigma \circ \sigma^*. \tag{3.5}$$

Putting together (3.4) and (3.5), we get

$$(f' \circ \varphi' \circ \sigma) \; R \; (g' \circ \varphi' \circ \sigma \circ \sigma^*),$$

which implies

$$f' \circ \varphi' = (f' \circ \varphi' \circ \sigma \circ \sigma^{-1}) \; R \; (g' \circ \varphi' \circ \sigma \circ \sigma^* \circ \sigma^{-1}),$$

as desired.

We show (3.2) assuming that α is a $\mathcal{K}_{\mathrm{p}}$-epimorphism, $\sigma^* \in \mathrm{Aut}(\mathbb{K})$. First, we observe that

$$(f \circ \varphi) \; R \; (g \circ \varphi \circ \sigma^*) \;\Rightarrow\; (f' \circ \alpha \circ \varphi) \, R \, (g' \circ \alpha \circ \varphi \circ \sigma^*). \tag{3.6}$$

Indeed, given $x \in \mathbb{K}$ let $a = \varphi(x)$, $b = \varphi(\sigma^*(x)) \in A$, so $f(a) R g(b)$. We find $r, s \in A$ and then $t \in A$, with

$$f(a) = g(r), \, rRs, \, g(s) = f(t), \, f(t) = g(b),$$

from which we get

$$f'(\alpha(a)) = g'(\alpha(r)) \, R \, g'(\alpha(s)) = f'(\alpha(t)) = g'(\alpha(b)).$$

To see (3.2), we repeat the argument for (3.1). This argument goes through since, by (3.6), we get (3.4) assuming $\sigma^* \in B^{(1)}_{f,g;\varphi}$, that is, $(f \circ \varphi) \, R \, (g \circ \varphi \circ \sigma^*)$.

Finally, towards (3.3) assuming α is a $\mathcal{K}_\mathrm{p}$-epimorphism, note that

$$(f \circ \varphi) \;=\; (g \circ \varphi \circ \sigma^*) \;\Rightarrow\; (f' \circ \alpha \circ \varphi) \;=\; (g' \circ \alpha \circ \varphi \circ \sigma^*). \tag{3.7}$$

Therefore, if $\sigma^* \in B^{(0)}_{f,g;\varphi}$, so $f \circ \varphi \;=\; g \circ \varphi \circ \sigma^*$, then

$$(f' \circ \alpha \circ \varphi) \;=\; (g' \circ \alpha \circ \varphi \circ \sigma^*). \tag{3.8}$$

Putting together (3.8) and (3.5) the same way implies

$$(f' \circ \varphi') \;=\; (g' \circ \varphi' \circ \sigma \circ \sigma^* \circ \sigma^{-1}),$$

as desired. $\square_{\text{Lemma3.6}}$

With some additional work one can show that in the lemma above, if α is an approximate $\mathcal{K}_\mathrm{p}$-epimorphism, then

$$\sigma B^{(d)}_{f,g;\varphi}\sigma^{-1} \subseteq B^{(2d+2)}_{f',g';\varphi'}, \;\text{ for all } d \in \mathbb{N}, \text{ and}$$
$$\sigma B^{(d)}_{f,g;\varphi}\sigma^{-1} \subseteq B^{(2d+1)}_{f',g';\varphi'}, \;\text{ if } d \text{ is even},$$

and if α is a $\mathcal{K}_\mathrm{p}$-epimorphism, then

$$\sigma B^{(d)}_{f,g;\varphi}\sigma^{-1} \subseteq B^{(d)}_{f',g';\varphi'}.$$

Lemma 3.7. *Given epimorphisms $\varphi_i, \psi_i \colon \mathbb{K} \to B_i$ with $i < n$, there exist objects (A, B, f_i, g_i) in $\mathcal{K}_p$, for $i < n$, and $\varphi \in \mathrm{Epi}(\mathbb{K}, A)$, such that*

$$B^{(1)}_{f_i,g_i;\varphi} \subseteq B^{(1)}_{\varphi_i,\psi_i} \;\text{ for all } i < n,$$

and

$$B^{(0)}_{f_i,g_i;\varphi} \subseteq B^{(0)}_{\varphi_i,\psi_i} \;\text{ for all } i < n,$$

Proof. We start with a claim.

Claim 3.8. *Given $\gamma_i \in \mathrm{Epi}(\mathbb{K}, B)$, for $i < m$, there exist $A \in \mathcal{K}$ and $h_i \in \mathrm{Epi}(A, B)$, and $\varphi \in \mathrm{Epi}(\mathbb{K}, A)$ such that*

$$\gamma_i = h_i \circ \varphi, \;\text{ for all } i < m.$$

Proof. We will use properties (A1) and (A3) of $\mathbb{K}$. Consider the function

$$\gamma_0 \times \cdots \times \gamma_{m-1} : \mathbb{K}^m \to B^m,$$

which is obviously continuous. By property (A1), there are $A \in \mathcal{K}$ and $\varphi \in \mathrm{Epi}(\mathbb{K}, A)$ such that $\gamma_0 \times \cdots \times \gamma_{m-1}$ factors through φ, that is, for each $x \in \mathbb{K}$, the tuple $(\gamma_0(x), \ldots, \gamma_{m-1}(x))$ depends only on $\varphi(x) \in A$. Let $h_i : A \to B$, for $i < m$, be defined by the equalities

$$h_i(\varphi(x)) = \gamma_i(x), \;\text{ for all } i < m.$$

By property (A3) of $\mathbb{K}$, $h_0, \ldots, h_{m-1}$ are epimorphisms. $\square_{\text{Claim3.8}}$

We proceed to proving the conclusion of the lemma. Directly from the definition of the sets $B^{(0)}_{f_i,g_i;\varphi}$ and $B^{(0)}_{\varphi_i,\psi_i}$ (and that of the sets $B^{(1)}_{f_i,g_i;\varphi}$ and $B^{(1)}_{\varphi_i,\psi_i}$), one sees that it suffices to find $A, B \in \mathcal{K}$, $\varphi \in \mathrm{Epi}(\mathbb{K}, A)$, $f_i, g_i \in \mathrm{Epi}(A, B)$, and $\beta_i \in \mathrm{Epi}(B, B_i)$ such that

$$\beta_i \circ f_i \circ \varphi = \varphi_i \;\text{ and }\; \beta_i \circ g_i \circ \varphi = \psi_i, \;\text{ for all } i < n. \tag{3.9}$$

By the joint projection property for $\mathcal{K}$, there exist $B \in \mathcal{K}$ and $\beta_i \in \mathrm{Epi}(B, B_i)$ for $i < n$. Now, since $\mathbb{K}$ is the projective Fraïssé limit of $\mathcal{K}$, there exist $\gamma_i, \xi_i \in \mathrm{Epi}(\mathbb{K}, B)$ such that

$$\beta_i \circ \gamma_i = \varphi_i \text{ and } \beta_i \circ \xi_i = \psi_i, \text{ for all } i < n. \tag{3.10}$$

Apply Claim 3.8 to the $m = 2n$ epimorphisms γ_i, ξ_i, $i < n$, obtaining $A \in \mathcal{K}$, $\varphi \in \mathrm{Epi}(\mathcal{K}, A)$, and $f_i, g_i \in \mathrm{Epi}(A, B)$ such that

$$\gamma_i = f_i \circ \varphi \text{ and } \xi_i = g_i \circ \varphi, \text{ for all } i < n. \tag{3.11}$$

Now (3.9) is implied by (3.10) and (3.11), and the lemma follows. $\square_{\text{Lemma 3.7}}$

Proof of Theorem 3.4. We first consider (i). Fix non-empty open sets $U, V \subseteq \mathrm{Homeo}(K)^n$. We need to show that

$$\sigma\, \overline{\gamma}\, \sigma^{-1} \in U, \text{ for some } \overline{\gamma} \in V \text{ and } \sigma \in \mathrm{Aut}(\mathbb{K}). \tag{3.12}$$

First, by our assumptions, $V \cap \mathrm{Aut}(\mathbb{K}) \neq \emptyset \neq V \cap \mathrm{Aut}(\mathbb{K})$.

By Corollary 2.5. and Lemma 3.7, we can assume that for some $(A, B, f_i, g_i) \in \mathcal{K}_{\mathrm{p}}$, $i < n$, and $\varphi \in \mathrm{Epi}(\mathbb{K}, A)$, we have

$$V \cap \mathrm{Aut}(\mathbb{K}) = \prod_{i<n} B^{(1)}_{f_i, g_i; \varphi},$$

and, similarly, for some $(A', B', f'_i, g'_i) \in \mathcal{K}_{\mathrm{p}}$, $i < n$, and $\varphi' \in \mathrm{Epi}(\mathbb{K}, A')$,

$$U \cap \mathrm{Aut}(\mathbb{K}) = \prod_{i<n} B^{(1)}_{f'_i, g'_i; \varphi'}.$$

By the approximate JPP, we obtain $A^+, B^+ \in \mathcal{K}$, $f_i^+, g_i^+ \in \mathrm{Epi}(A^+, B^+)$, $i < n$, and $\mathcal{K}$-epimorphisms $\alpha : A^+ \to A$, $\alpha' : A^+ \to A'$, such that, for each $i < n$, α and α' are approximate $\mathcal{K}_{\mathrm{p}}$-epimorphisms from (A^+, B^+, f_i^+, g_i^+) to (A, B, f_i, g_i) and to (A', B', f'_i, g'_i), respectively.

Pick $\varphi^+ \in \mathrm{Epi}(\mathbb{K}, A^+)$ such that $\varphi = \alpha \circ \varphi^+$. Then, from the definitions of the two sets in (3.13) and from α being an approximate $\mathcal{K}_p$-epimorphism, we get

$$B^{(0)}_{f_i^+, g_i^+; \varphi^+} \subseteq B^{(1)}_{f_i, g_i; \varphi}, \text{ for all } i < n. \tag{3.13}$$

Pick $\sigma \in \mathrm{Aut}(\mathbb{K})$ with $\sigma \in B_{\alpha' \circ \varphi^+, \varphi'}$. By Lemma 3.6, we have

$$\sigma(B^{(0)}_{f_i^+, g_i^+; \varphi^+})\sigma^{-1} \subseteq B^{(1)}_{f'_i, g'_i; \varphi'}, \text{ for all } i < n. \tag{3.14}$$

Now, (3.12) follows from (3.13) and (3.14) since the set $B^{(0)}_{f_i^+, g_i^+; \varphi^+}$ is non-empty.

We turn to (ii). For the non-empty open set $U \subseteq \mathrm{Homeo}(K)^n$ there again exist $(A', B', f'_i, g'_i) \in \mathcal{K}_{\mathrm{p}}$, $i < n$, and $\varphi' \in \mathrm{Epi}(\mathbb{K}, A')$ with

$$U \cap \mathrm{Aut}(\mathbb{K}) = \prod_{i<n} B^{(1)}_{f'_i, g'_i; \varphi'}.$$

Fix a non-empty open set $V \subseteq \mathrm{Aut}(\mathbb{K})$. By Corollary 2.5. and Lemma 3.7, we can assume that for some $(A, B, f_i, g_i) \in \mathcal{K}_{\mathrm{p}}$, $i < n$, and $\varphi \in \mathrm{Epi}(\mathbb{K}, A)$, we have

$$V = \prod_{i<n} B^{(0)}_{f_i, g_i; \varphi}.$$

So again it remains to show that

$$\sigma\,\overline{\gamma}\,\sigma^{-1} \in U, \quad \text{for some } \overline{\gamma} \in V \text{ and } \sigma \in \mathrm{Aut}(\mathbb{K}). \tag{3.15}$$

By the half-approximate JPP, we obtain $A^+, B^+ \in \mathcal{K}$, $f_i^+, g_i^+ \in \mathrm{Epi}(A^+, B^+)$, $i < n$, and $\mathcal{K}$-epimorphisms $\alpha : A^+ \to A$, $\alpha' : A^+ \to A'$, such that, for each $i < n$, α is a $\mathcal{K}_{\mathrm{p}}$-epimorphism from (A^+, B^+, f_i^+, g_i^+) to (A, B, f_i, g_i) and α' is an approximate $\mathcal{K}_{\mathrm{p}}$-epimorphism to (A', B', f_i', g_i').

If $\varphi^+ \in \mathrm{Epi}(\mathbb{K}, A^+)$ is such that $\varphi = \alpha \circ \varphi^+$, then we can proceed as in the proof of (i), so

$$B^{(0)}_{f_i^+, g_i^+; \varphi^+} \subseteq B^{(0)}_{f_i, g_i; \varphi}, \quad \text{for all } i < n,$$

and if $\sigma \in B_{\alpha' \circ \varphi^+, \varphi'}$, then

$$\sigma(B^{(0)}_{f_i^+, g_i^+; \varphi^+})\sigma^{-1} \subseteq B^{(1)}_{f_i', g_i'; \varphi'}, \quad \text{for all } i < n.$$

The proof of (iii) is the same. We start from

$$U = \prod_{i<n} B^{(0)}_{f_i', g_i'; \varphi'},$$

and

$$V = \prod_{i<n} B^{(0)}_{f_i, g_i; \varphi},$$

for each $i < n$, α is a $\mathcal{K}_{\mathrm{p}}$-epimorphism from (A^+, B^+, f_i^+, g_i^+) to (A, B, f_i, g_i) as well as α' is a $\mathcal{K}_{\mathrm{p}}$-epimorphism to (A', B', f_i', g_i'). So if $\varphi^+ \in \mathrm{Epi}(\mathbb{K}, A^+)$ is such that $\varphi = \alpha \circ \varphi^+$, then

$$B^{(0)}_{f_i^+, g_i^+; \varphi^+} \subseteq B^{(0)}_{f_i, g_i; \varphi}, \quad \text{for all } i < n,$$

and if $\sigma \in B_{\alpha' \circ \varphi^+, \varphi'}$, then

$$\sigma(B^{(0)}_{f_i^+, g_i^+; \varphi^+})\sigma^{-1} \subseteq B^{(0)}_{f_i', g_i'; \varphi'}, \quad \text{for all } i < n.$$

$\square_{\text{Theorem3.4}}$

The following corollary is an immediate consequence of Theorem 3.4.

Corollary 3.9. *Suppose that $\mathcal{K}$ is a transitive projective Fraïssé class with the property that* Aut($\mathbb{K}$) *has a dense image under the canonical homomorphism into* Homeo(K)*, let $n \in \mathbb{N}$.*

(i) If $\mathcal{K}_p^{\times n}$ has the approximate JPP, then

$$\{\overline{\gamma} \in \mathrm{Homeo}(K)^n : \ \overline{\gamma}^{\mathrm{Aut}(\mathbb{K})} \textit{ is dense}\} \ \textit{ is dense } G_\delta \textit{ in } \mathrm{Homeo}(K)^n.$$

(ii) If $\mathcal{K}_p^{\times n}$ has the half-approximate JPP, then

$$\{\overline{\gamma} \in \mathrm{Aut}(\mathbb{K})^n : \ \overline{\gamma}^{\mathrm{Aut}(\mathbb{K})} \textit{ is dense in } \mathrm{Homeo}(K)^n\} \ \textit{ is dense } G_\delta \textit{ in } \mathrm{Aut}(\mathbb{K})^n.$$

(iii) If $\mathcal{K}_p^{\times n}$ has the JPP, then

$$\{\overline{\gamma} \in \mathrm{Aut}(\mathbb{K})^n : \ \overline{\gamma}^{\mathrm{Aut}(\mathbb{K})} \textit{ is dense in } \mathrm{Aut}(\mathbb{K})^n\} \ \textit{ is dense } G_\delta \textit{ in } \mathrm{Aut}(\mathbb{K})^n.$$

Corollary 3.9 implies the following.

Corollary 3.10. *Suppose that $\mathcal{K}$ is a transitive projective Fraïssé class with the property that* $\mathrm{Aut}(\mathbb{K})$ *has a dense image under the canonical homomorphism into* $\mathrm{Homeo}(K)$.

(*i*) *If $\mathcal{K}_p^{\times n}$ has the approximate JPP for each $n \in \mathbb{N}$, then*
$$\{\overline{\gamma} \in \mathrm{Homeo}(K)^{\mathbb{N}} :\ \overline{\gamma}^{\mathrm{Aut}(\mathbb{K})} \textit{ is dense}\} \textit{ is dense } G_\delta \textit{ in } \mathrm{Homeo}(K)^{\mathbb{N}}.$$
(*ii*) *If $\mathcal{K}_p^{\times n}$ has the half-approximate JPP for each $n \in \mathbb{N}$, then*
$$\{\overline{\gamma} \in \mathrm{Aut}(\mathbb{K})^{\mathbb{N}} :\ \overline{\gamma}^{\mathrm{Aut}(\mathbb{K})} \textit{ is dense in } \mathrm{Homeo}(K)^{\mathbb{N}}\} \textit{ is dense } G_\delta \textit{ in } \mathrm{Aut}(\mathbb{K})^{\mathbb{N}}.$$
(*iii*) *If $\mathcal{K}_p^{\times n}$ has the JPP for each $n \in \mathbb{N}$, then*
$$\{\overline{\gamma} \in \mathrm{Aut}(\mathbb{K})^{\mathbb{N}} :\ \overline{\gamma}^{\mathrm{Aut}(\mathbb{K})} \textit{ is dense in } \mathrm{Aut}(\mathbb{K})^{\mathbb{N}}\} \textit{ is dense } G_\delta \textit{ in } \mathrm{Aut}(\mathbb{K})^{\mathbb{N}}.$$

Proof. We write the proof only for (ii), as (i) and (iii) follow by the same argument. Note that $\overline{\gamma} = (\gamma_i)_{i\in\mathbb{N}} \in \mathrm{Aut}(\mathbb{K})^{\mathbb{N}}$ is such that $\overline{\gamma}^{\mathrm{Aut}(\mathbb{K})}$ is dense in $\mathrm{Homeo}(K)^{\mathbb{N}}$ precisely when for each $n \in \mathbb{N}$, $n \geq 1$, the orbit of the finite tuple $(\gamma_i)_{i<n}$ under the diagonal conjugacy action of $\mathrm{Aut}(\mathbb{K})$ is dense in $\mathrm{Homeo}(K)^n$. Now, the conclusion of (ii) follows from Corollary 3.9 (ii) since if $G \subseteq \mathrm{Aut}(\mathbb{K})^n$ is a dense G_δ in $\mathrm{Aut}(\mathbb{K})^n$, then the set
$$\{(\gamma_i)_{i\in\mathbb{N}} \in \mathrm{Aut}(\mathbb{K})^{\mathbb{N}} \mid (\gamma_i)_{i<n} \in G\}$$
is a dense G_δ in $\mathrm{Aut}(\mathbb{K})^{\mathbb{N}}$. $\square_{\text{Corollary 3.10}}$

Now we state and prove the converse to Theorem 3.4. .

Theorem 3.11. *Suppose that $\mathcal{K}$ is a transitive projective Fraïssé class with the property that* $\mathrm{Aut}(\mathbb{K})$ *has a dense image under the canonical homomorphism into* $\mathrm{Homeo}(K)$, *and let $n \in \mathbb{N}$.*

(*i*) *If there exists $\bar{\gamma} \in \mathrm{Homeo}(K)^n$ such that $\bar{\gamma}^{\mathrm{Homeo}(K)}$ is dense in $\mathrm{Homeo}(K)^n$, then $\mathcal{K}_p^{\times n}$ has the approximate JPP.*
(*ii*) *If the set of all $\bar{\gamma} \in \mathrm{Aut}(\mathbb{K})^n$ such that $\bar{\gamma}^{\mathrm{Homeo}(K)}$ is dense in $\mathrm{Homeo}(K)^n$ is comeager in $\mathrm{Aut}(\mathbb{K})^n$, then $\mathcal{K}_p^{\times n}$ has the half-approximate JPP.*
(*iii*) *If there exists $\bar{\gamma} \in \mathrm{Aut}(\mathbb{K})^n$ such that $\bar{\gamma}^{\mathrm{Aut}(\mathbb{K})}$ is dense in $\mathrm{Aut}(\mathbb{K})^n$, then $\mathcal{K}_p^{\times n}$ has the JPP.*

Proof. For simplicity we assume that $n = 1$. First we consider (*i*).

Fix (A, B, f, g), $(A', B', f', g') \in \mathcal{K}_{\mathrm{p}}$. Pick $\varphi \in \mathrm{Epi}(\mathbb{K}, A)$, $\varphi' \in \mathrm{Epi}(\mathbb{K}, A')$. By Corollary 2.5 $B^{(1)}_{f,g;\varphi} \subseteq \mathrm{Aut}(\mathbb{K})$ has non-empty interior with respect to the topology inherited from $\mathrm{Homeo}(K)$, so for some nonempty open set $U \subseteq \mathrm{Homeo}(K)$ we have
$$U \cap \mathrm{Aut}(\mathbb{K}) \subseteq B^{(1)}_{f,g;\varphi}.$$
Similarly, for some non-empty open set U' in $\mathrm{Homeo}(K)$ we have
$$U' \cap \mathrm{Aut}(\mathbb{K}) \subseteq B^{(1)}_{f',g';\varphi'}. \tag{3.16}$$

By density of $\mathrm{Aut}(\mathbb{K})$ in $\mathrm{Homeo}(K)$ and continuity we can infer the following.

Claim 3.12. *There exist $\sigma, \sigma', \tau \in$ Aut$(\mathbb{K})$, such that*
$$\tau\sigma\tau^{-1} = \sigma', \tag{3.17}$$

moreover,

$$\sigma \in B^{(1)}_{f,g;\varphi}, \tag{3.18}$$

$$\sigma' \in B^{(1)}_{f',g';\varphi'}. \tag{3.19}$$

Proof. By our assumptions there are $\varsigma \in U$, $\varsigma' \in U'$ that are conjugate to each other, i.e. for some $\tau_0 \in \mathrm{Homeo}(K)$

$$\tau_0 \varsigma \tau_0^{-1} = \varsigma' \in U'.$$

By continuity of the composition and inverse, taking $\tau \in \mathrm{Aut}(\mathbb{K})$ close enough to τ_0, $\sigma \in \mathrm{Aut}(\mathbb{K})$ close enough to ς (and recalling (3.16)) we will have

$$\tau\sigma\tau^{-1} \in U' \cap \mathrm{Aut}(\mathbb{K}),$$

therefore setting $\sigma' = \tau\sigma\tau^{-1}$ works. $\square_{\text{Claim3.12}}$

We are going to find $A^+, B^+ \in \mathcal{K}$, $f^+, g^+ \in \mathrm{Epi}(A^+, B^+)$, $\varphi^+ \in \mathrm{Epi}(\mathbb{K}, A^+)$ such that

$$\sigma \in B_{f^+,g^+;\varphi^+},$$

and φ, $\varphi' \circ \tau$ factor through $g^+ \circ \varphi^+$ (therefore they factor through φ^+, too). This can be done by picking $\psi^* \in \mathrm{Epi}(\mathbb{K}, C)$ such that $(\varphi \times \varphi' \circ \tau)$ factors through ψ^*, and applying Lemma 3.8 to $\varphi^* = \psi^* \circ \sigma$ and ψ^*. If $f^+ \circ \varphi^+ = \psi^* \circ \sigma^{-1}$ and $g^+ \circ \varphi^+ = \psi^*$, then as φ, $\varphi' \circ \tau$ factor through ψ^*, i.e. $\varphi = \delta \circ \psi^*$, $\varphi' \circ \tau = \delta' \circ \psi^*$ for some δ and δ', clearly

$$\delta \circ g^+ \circ \varphi^+ = \delta \circ \psi^* = \varphi. \tag{3.20}$$

$$\delta' \circ g^+ \circ \varphi^+ = \delta' \circ \psi^* = \varphi' \circ \tau. \tag{3.21}$$

Moreover, it is not difficult to see that that $B_{\varphi^*,\psi^*} = B_{\psi^* \circ \sigma, \psi^*} \ni \sigma$.

We let $\alpha = \delta \circ g^+ : A^+ \to A$ so that $\varphi = \alpha \circ \varphi^+$, and let $\alpha' = \delta' \circ g^+ : A^+ \to A'$, so that $\varphi' \circ \tau = \alpha' \circ \varphi^+$. To finish the proof of the theorem, it suffices to show that $\alpha \in \mathrm{Epi}(A^+, A)$, $\alpha' \in \mathrm{Epi}(A^+, A')$, moreover, they are approximate epimorphisms between the respective objects in $\mathcal{K}_{\mathrm{p}}$. First, it is easy to check that α is a strong homomorphism, so by property (A3) $\alpha \in \mathrm{Epi}(A^+, A)$, and similarly, $\varphi' \circ \tau$ and φ^+ are epimorphisms, so are α'.

First we check that α' is an approximate epimorphism. Let $a_0, a_1 \in A^+$ with $f^+(a_0) = g^+(a_1)$, we need to argue that $f'(\alpha'(a_0)) \; R \; g'(\alpha'(a_1))$. Pick $x \in \mathbb{K}$ with $\varphi^+(\tau^{-1}(x)) = a_0$.

Claim 3.13. *Suppose that* $\tau \in \mathrm{Aut}(\mathbb{K})$*,* δ'*,* g^+*,* f^+*,* α' *are* $\mathcal{K}$*-epimorphisms,* $\varphi^+ \in \mathrm{Epi}_{\mathbb{K}}$ *that satisfy*

$$\delta' \circ g^+ = \alpha',$$

and $\sigma \in B_{f^+,g^+;\varphi^+}$*. If* a_0, a_1 *are such that* $f^+(a_0) = g^+(a_1)$ *and* $\varphi^+(\tau^{-1}(x)) = a_0$*, then*

$$\alpha'(a_1) = \alpha'(\varphi^+(\sigma\tau^{-1}(x)).$$

Proof. Indeed, $\sigma \in B_{f^+,g^+;\varphi^+}$ implies $f^+(\varphi^+(\tau^{-1}(x))) = g^+(\varphi^+(\sigma(\tau^{-1}(x))))$, using $\varphi^+(\tau^{-1}(x)) = a_0$ we get

$$g^+(\varphi^+(\sigma(\tau^{-1}(x)))) = f^+(a_0) = g^+(a_1).$$

So applying δ', and using $\alpha' = \delta' \circ g^+$

$$\alpha' \circ \varphi^+(\sigma\tau^{-1}(x)) = \alpha'(a_1).$$

$\square_{\text{Claim3.13}}$

So pick $x \in \mathbb{K}$ with $\varphi^+(\tau^{-1}(x)) = a_0$ and note that by the claim the condition $f'(\alpha'(a_0)) \; R \; g'(\alpha'(a_1))$ is equivalent to

$$f'(\alpha'(\varphi^+(\tau^{-1}(x)))) \; R \; g'(\alpha'(\varphi^+(\sigma\tau^{-1}(x)))),$$

so we need to check this for all x, i.e.

$$(f' \circ \alpha' \circ \varphi^+ \circ \tau^{-1}) \; R \; (g' \circ \alpha' \circ \varphi^+ \circ \sigma\tau^{-1}).$$

But using $\alpha' \circ \varphi^+ = \varphi' \circ \tau$, this is equivalent to

$$(f' \circ \varphi' \circ \tau \circ \tau^{-1} \; R \; g' \circ \varphi' \circ \tau \circ \sigma \circ \tau^{-1}).$$

We obtained that

$$(f' \circ \varphi' \circ \tau \circ \tau^{-1} \; R \; g' \circ \varphi' \circ \tau \circ \sigma \circ \tau^{-1}) \;\Rightarrow\; \alpha' \text{ is an approximate epimorphism} \tag{3.22}$$

. But the premise is true, since $\sigma' = \tau\sigma\tau^{-1} \in B^{(1)}_{f',g';\varphi'}$, we are done. It remains to show that α is an approximate epimorphism.

Claim 3.14. *Suppose that δ, g^+, f^+, α are $\mathcal{K}$-epimorphisms, $\varphi^+ \in \mathrm{Epi}_{\mathbb{K}}$ that satisfy*

$$\delta \circ g^+ = \alpha,$$

and $\sigma \in B_{f^+,g^+;\varphi^+}$. If a_0, a_1 are such that $f^+(a_0) = g^+(a_1)$ and $\varphi^+(x) = a_0$, then

$$\alpha(a_1) = \alpha(\varphi^+(\sigma(x)).$$

Proof. Indeed, $\sigma \in B_{f^+,g^+;\varphi^+}$ implies $f^+(\varphi^+(x)) = g^+(\varphi^+(\sigma(x)))$, using $\varphi^+(x) = a_0$ we get

$$g^+(\varphi^+(\sigma(x))) = f^+(a_0) = g^+(a_1).$$

Now applying δ, and using $\alpha = \delta \circ g^+$

$$\alpha \circ \varphi^+(\sigma(x)) = \alpha(a_1).$$

$\square_{\text{Claim3.14}}$

Let $a_0, a_1 \in A^+$ be such that $f^+(a_0) = g^+(a_1)$. We need to argue that we have $f(\alpha(a_0)) \; R \; g(\alpha(a_1))$. Pick $x \in \mathbb{K}$ with $\varphi^+((x)) = a_0$.

The claim gives that $f(\alpha(a_0)) \; R \; g(\alpha(a_1))$ is equivalent to

$$f(\alpha(\varphi^+(x))) \; R \; g(\alpha(\varphi^+(\sigma(x))),$$

so we need to verify that this holds for all x, i.e.

$$(f \circ \alpha \circ \varphi^+) \; R \; (g \circ \alpha \circ \varphi^+ \circ \sigma).$$

But using $\alpha \circ \varphi^+ = \varphi$, this is equivalent to

$$(f \circ \varphi) \; R \; (g \circ \varphi \circ \sigma).$$

So we got that

$$(f \circ \varphi)\ R\ (g \circ \varphi \circ \sigma)\ \Rightarrow\ \alpha \text{ is an approximate } \mathcal{K}_{\mathrm{p}}\text{-epimorphism.} \tag{3.23}$$

But the premise is true, since $\sigma \in B^{(1)}_{f,g;\varphi}$, we are done.

Now we sketch the proof of (*ii*).

Fix again (A, B, f, g), $(A', B', f', g') \in \mathcal{K}_{\mathrm{p}}$, and pick $\varphi \in \mathrm{Epi}(\mathbb{K}, A)$, $\varphi' \in \mathrm{Epi}(\mathbb{K}, A')$. By Corollary 2.5 for some nonempty open set $U \subseteq \mathrm{Aut}(\mathbb{K})$ we have

$$U \subseteq B^{(0)}_{f,g;\varphi},$$

and for some non-empty open set U' in $\mathrm{Homeo}(K)$ we have

$$U' \cap \mathrm{Aut}(\mathbb{K}) \subseteq B^{(1)}_{f',g';\varphi'}. \tag{3.24}$$

Then we find $\sigma, \sigma', \tau \in \mathrm{Aut}(\mathbb{K})$, such that

- $\tau\sigma\tau^{-1} = \sigma'$,
- $\sigma \in B^{(0)}_{f,g;\varphi}$,
- $\sigma' \in B^{(1)}_{f',g';\varphi'}$.

Next we find $A^+, B^+ \in \mathcal{K}$, $f^+, g^+ \in \mathrm{Epi}(A^+, B^+)$, $\varphi^+ \in \mathrm{Epi}(\mathbb{K}, A^+)$ such that

$$\sigma \in B_{f^+,g^+;\varphi^+},$$

and δ and δ' with

$$\delta \circ g^+ \circ \varphi^+ = \varphi. \tag{3.25}$$

$$\delta' \circ g^+ \circ \varphi^+ = \varphi' \circ \tau. \tag{3.26}$$

We let $\alpha = \delta \circ g^+ : A^+ \to A$ so that $\varphi = \alpha \circ \varphi^+$, and let $\alpha' = \delta' \circ g^+ : A^+ \to A'$, so that $\varphi' \circ \tau = \alpha' \circ \varphi^+$. The same argument as above shows that α' is an approximate epimorphism between (A^+, B^+, f^+, g^+) and (A', B', f', g').

It remains to show that α is an epimorphism between (A^+, B^+, f^+, g^+) and (A, B, f, g). This case Claim 3.14 still applies, and modifying the argument following the claim one can get

$$(f \circ \varphi)\ =\ (g \circ \varphi \circ \sigma)\ \Rightarrow\ \alpha \text{ is a } \mathcal{K}_{\mathrm{p}}\text{-epimorphism.} \tag{3.27}$$

But the premise is true, since $\sigma \in B^{(0)}_{f,g;\varphi}$, we are done.

To get (*iii*) we need to make minor changes to the proof of (*ii*). We have

$$U \subseteq B^{(0)}_{f,g;\varphi},$$

and

$$U' = B^{(0)}_{f',g';\varphi'},$$

and then we find $\sigma, \sigma', \tau \in \mathrm{Aut}(\mathbb{K})$, such that

- $\tau\sigma\tau^{-1} = \sigma'$,
- $\sigma \in B^{(0)}_{f,g;\varphi}$,
- $\sigma' \in B^{(0)}_{f',g';\varphi'}$.

Constructing again α and α', one shows (3.27), and

$$(f' \circ \varphi' \circ \tau \circ \tau^{-1}\ =\ g' \circ \varphi' \circ \tau \circ \sigma \circ \tau^{-1})\ \Rightarrow\ \alpha' \text{ is a } \mathcal{K}_{\mathrm{p}}\text{-epimorphism.}$$

$\square_{\text{Theorem3.11}}$

§ 4. Dense conjugacy classes in products—the pseudoarc case

We refer the reader to Section 1 for the definition and relevant properties of $\mathcal{P}$—the category of finite reflexive linear graphs. By $\mathbb{P}$ we denote the projective Fraïssé limit of $\mathcal{P}$. In this case, the canonical continuous homomorphism $\mathrm{pr} : \mathrm{Aut}(\mathbb{P}) \to \mathrm{Homeo}(P)$ is injective, and we will identify $\mathrm{Aut}(\mathbb{P})$ with its image, that is,

$$\mathrm{Aut}(\mathbb{P}) \leq \mathrm{Homeo}(P).$$

The main theorem in this section is the following application of Theorem 3.4. Note that it implies Theorem 1.1 from the introduction.

Theorem 4.1. *(i) For every $n \in \mathbb{N}$, the set of all $\bar{\gamma} \in \mathrm{Aut}(\mathbb{P})^n$ such that the orbit of $\bar{\gamma}$ under the diagonal conjugacy action by $\mathrm{Aut}(\mathbb{P})$ is dense in $\mathrm{Homeo}(P)^n$ is comeager in $\mathrm{Aut}(\mathbb{P})^n$.*

(ii) The set of all $\bar{\gamma} \in \mathrm{Aut}(\mathbb{P})^{\mathbb{N}}$ such that the orbit of $\bar{\gamma}$ under the diagonal conjugacy action by $\mathrm{Aut}(\mathbb{P})$ is dense in $\mathrm{Homeo}(P)^{\mathbb{N}}$ is comeager in $\mathrm{Aut}(\mathbb{P})^{\mathbb{N}}$.

The key to the proof of the theorem above is the following lemma.

Lemma 4.2. *Suppose that $A, B, A', B' \in \mathcal{P}$, there exist $A^+, B^+ \in \mathcal{P}$ and $\alpha \in \mathrm{Epi}(A^+, A)$, $\alpha' \in \mathrm{Epi}(A^+, A')$, $\beta \in \mathrm{Epi}(B^+, B)$, and $\beta' \in \mathrm{Epi}(B^+, B')$ with the following property:*
for all $f \in \mathrm{Epi}(A, B)$, $f' \in \mathrm{Epi}(A', B')$, there exists $f^+ \in \mathrm{Epi}(A^+, B^+)$ such that

$$f \circ \alpha = \beta \circ f^+ \quad \text{and} \quad (f' \circ \alpha')\ R\ (\beta' \circ f^+).$$

The following corollary is an immediate consequence of Lemma 4.2.

Corollary 4.3. *Suppose that $A, B, A', B' \in \mathcal{P}$, f_i, g_i, f'_i, g'_i, $i = 1, \ldots, n$, are such that (A, B, f_i, g_i), $(A', B', f'_i, g'_i) \in \mathcal{P}_p$. Then, for some A^+, B^+, f_i^+, g_i^+, with $1 \leq i \leq n$, and $\alpha : A^+ \to A$, $\alpha' : A^+ \to A'$, we have that, for each $1 \leq i \leq n$,*

- *the map α is an epimorphism between (A^+, B^+, f_i^+, g_i^+) and (A, B, f_i, g_i),*
- *the map α' is an approximate epimorphism between (A^+, B^+, f_i^+, g_i^+) and (A', B', f'_i, g'_i).*

Proof of Lemma 4.2. By possibly extending A with an exact epimorphism, we can assume that

$(*)_1$ $f^{-1}(b)$ is the union of intervals each is of length at least 2,

since exact epimorphisms are closed under composition. We introduce the notation $\mathbf{R}(.,.)$ for division with reminder, and write $\mathbf{R}(j, m) = k$ iff $k \in [0, m)$ and $j \equiv k \mod m$.

First we define A^+ and B^+, we start with the latter. Let $m = |B|$, $m' = |B'|$, that is, B is a path of length m, B' is a path of length m'. We set B^+ to be a path of length mm'. Identifying B' with $[0, m')$ and B^+ with $[0, mm')$, define $\beta' : B^+ \to B'$ by the equation

$(*)_2$ $\beta'(j) = \lfloor j/m \rfloor$.

Next we define $\beta : B^+ \to B$, which will satisfy that whenever $\mathbf{R}(j, m) = k$, then

$$\beta(j) \in \{k, m - 1 - k\}.$$

The epimorphism β is given by

$(*)_3$ $\beta(j) = \mathbf{R}(j,m)$, for $j \in [2lm,(2l+1)m)$ with $0 \le 2l \le m'-1$;
$(*)_4$ $\beta(j) = m-1-\mathbf{R}(j,m)$, for $j \in [(2l+1)m,(2l+2)m)$ with $0 \le 2l \le m'-2$.

The reflexive linear graph A^+ and the epimorphisms α, α' are defined similarly. If $n = |A|$, $n' = |A'|$, then let A^+ to be a path of length nn'. Define $\alpha' : A^+ \to A'$ by the equation

$(*)_5$ $\alpha'(j) = \lfloor j/n \rfloor$.

The map $\alpha : A^+ \to A$ is defined by

$(*)_6$ $\alpha(j) = \mathbf{R}(j,n)$, for $j \in [2ln,(2l+1)n)$ with $0 \le 2l \le n'-1$,
$(*)_7$ $\alpha(j) = n-1-\mathbf{R}(j,n)$, for $j \in [(2l+1)n,(2l+2)n)$ with $0 \le 2l \le n'-2$.

Fix $f \in \mathrm{Epi}(A,B)$ and $f' \in \mathrm{Epi}(A',B')$. We are going to construct $f^+ \in \mathrm{Epi}(A^+,B^+)$ such that

$$\beta \circ f^+ \;=\; f \circ \alpha, \tag{4.1}$$

$$\beta' \circ f^+ \; R \; f' \circ \alpha'. \tag{4.2}$$

We note that if $J \subseteq A^+$ is an interval satisfying $f^+[J] \subseteq [lm,(l+1)m)$, for some $l < m'$, then $f^+ \restriction J$ is determined by f (by the requirement (4.1) and by injectivity of β on $[lm,(l+1)m)$ guaranteed by $(*)_3$, $(*)_4$).

We will define the sequence $s_0, s_1, \ldots, s_{n'-1}$ so that

(I) $s_j \in [jn,(j+1)n-1)$,
(II) for $a < n'n$,

$$\beta \circ f^+(a) = f \circ \alpha(a),$$

(III) if $j < n'-1$, then, for $a \in [jn, s_j]$,

$$\beta' \circ f^+(a) = f' \circ \alpha'(a),$$

and for $a \in [s_j+1,(j+1)n)$,

$$\beta' \circ f^+(a) = f'(\alpha'(a)+1),$$

moreover, for $j = n'-1$ and $a \in [jn,(j+1)n)$,

$$\beta' \circ f^+(a) = f' \circ \alpha'(a).$$

We note that (II), (III) imply (4.2), since $f'(\alpha'(a)) R f'(\alpha'(a)+1)$ because f' is an epimorphism. So it remains to prove the following statement.

Suppose that $j = 0$, or $j > 0$ and $s_0, s_1, \ldots, s_{j-1}$, $f^+ \restriction [0,jn)$ are defined so that (II), (III) hold. Then it is possible to define s_j and $f^+ \restriction [0,(j+1)n)$ so that they similarly satisfy (II), (III).

The statement is proved by induction on j. First we pick $f^+(jn) \in B^+$.

If $j = 0$, then we use that $\beta \times \beta'$ is a bijection between B^+ and $B \times B'$ and let $f^+(0)$ be the unique value in B^+ that satisfies

$$\beta' \circ f^+(0) = f' \circ \alpha'(0) \;\text{ and }\; \beta \circ f^+(0) = f \circ \alpha(0).$$

If $j > 0$, then by $((III))$ we have

$$\beta' \circ f^+(jn-1) = f' \circ (\alpha'(jn-1)+1),$$

and by the definition of α' we have $\alpha'(jn-1)+1 = j-1+1 = \alpha'(jn)$. Moreover, the way we defined α implies $\alpha(jn-1) = \alpha(jn)$, therefore letting $f^+(jn) = f^+(jn-1)$ will ensure (4.1) and (II). If $j < n'-1$, then consider the value

$$c = f'(\alpha'(jn)+1) = f'(\alpha'((j+1)n)) = f'(j+1).$$

If $c = f'(\alpha'(jn))$, or $j = n'-1$ and c is not defined then we just define $f^+\restriction[jn,(j+1)n)$ so that $\beta' \circ f^+\restriction[jn,(j+1)n) = c$, and $\beta \circ f^+\restriction[jn,(j+1)n) = f\circ\alpha\restriction[jn,(j+1)n)$. (In this case s_j can be just jn.)

Otherwise, $j < n'-1$ and $c \in \{f'(\alpha'(jn))-1, f'(\alpha'(jn))+1\}$ hold necessarily, and our aim is to ultimately get $\beta'(f^+((j+1)n-1)) = c$. For simplicity we assume that

$$c = f'(\alpha'(jn))+1 \tag{4.3}$$

and

$$f'(\alpha'(jn))(= \beta'(f^+(jn))) \text{ is even,} \tag{4.4}$$

the other three cases are handled the exact same way. By these assumptions and by the way β' was defined $((*)_2)$ we note that

$$(\beta')^{-1}(f'(\alpha'(jn)) = [f'(\alpha'(jn)n, (f'(\alpha'(jn))+1)n),$$

and

$$(\beta')^{-1}(f'(\alpha'(jn)+1) = [(f'(\alpha'(jn)+1)n, (f'(\alpha'(jn))+2)n).$$

Now we let $s_j \geq jn$ be minimal such that $f(\alpha(s_j)) = m-1$, and define $f^+\restriction[jn,s_j]$ so that $\beta' \circ f^+\restriction[jn,s_j] \equiv f'(\alpha'(jn))$, and $\beta \circ f^+\restriction[jn,s_j] = f\circ\alpha\restriction[jn,s_j]$. It follows that $f^+(s_j) = f'(\alpha'(jn)+1)n-1$. Using the assumption $(*)_1$ on f we obtain that $s_j+1 < (j+1)n$. Let $f^+(s_j+1) = f^+(s_j)+1$, using $(*)_3$

$$\beta(f^+(s_j+1)) = \beta(f^+(s_j)+1) = \beta(f'(\alpha'(jn)+1)n-1+1) = \beta(f'(\alpha'(jn)+1)n),$$

and note that $f'(\alpha'(jn)+1) = c = f'(\alpha'(jn))+1$ is odd by our assumption (4.4), therefore (by $(*)_4$)

$$\beta(f^+(s_j+1)) = \beta(f'(\alpha'(jn)+1)n) = m-1,$$

so $\beta(f^+(s_j+1)) = f(\alpha(s_j))$. Using that $s_j \geq jn$ is minimal with the condition $f(\alpha(s_j)) = m-1$ and recalling $(*)_1$, $(*)_6$ we get $s_j < (j+1)n-1$,

$$f(\alpha(s_j+1)) = f(\alpha(s_j)+1) = f(\alpha(s_j)) = m-1,$$

hence (II) remains true. It is easy to check that $\alpha'(s_j) = \alpha'(s_j+1)$ since $jn+1 \leq s_j+1 < (j+1)n$ and so

$$\beta'(f^+(s_j+1)) = \beta'(f^+(s_j))+1 = f'(\alpha'(s_j))+1 = f'(\alpha'(s_j+1))+1,$$

and we get $((III))$.

Finally, we define $f^+\restriction[s_j+1,(j+1)n)$ so that $\beta' \circ f^+\restriction[s_j+1,(j+1)n) \equiv \beta'(f^+(s_j+1))$, and $\beta \circ f^+\restriction[s_j+1,(j+1)n) = f\circ\alpha\restriction[s_j+1,(j+1)n)$.

The statement is proved and the lemma follows. $\square_{\text{Lemma4.2}}$

Proof of Theorem 4.1. (i) follows from Corollary 4.3 and Corollary 3.9.

(ii) follows from Corollary 4.3 and Corollary 3.10. $\square_{\text{Theorem4.1}}$

§ 5. A homeomorphism of the pseudoarc that is not conjugate to an automorphism of the pre-pseudoarc

In this section will concern $\mathcal{P}$, the class of finite connected linear graphs with all edge-preserving surjective homomorphisms (note that if a graph-homomorphism between two finite linear graphs is onto it is automatically surjective on the edges).

Theorem 5.1. *Let P be the pseudoarc, which we identify with $\mathrm{pr}[\mathbb{P}]$, the natural projection of the projective Fraïssé limit of the class of finite linear graphs. There exists a homeomorphism $\sigma \in \mathrm{Homeo}(P)$ that is not conjugate to any $\tau \in \mathrm{Aut}(\mathbb{P})$.*

The following is a key lemma; it provides a sufficient condition for a homeomorphism to not be conjugate to any automorphism of $\mathbb{P}$. Our argument hinges on the rigidity of automorphisms, that they preserve the algebra of regular open sets associated with the projections onto finite linear graphs. Our example that will be constructed in Proposition 5.4 will satisfy the premises.

Lemma 5.2. *Suppose that $\sigma : P \to P$ is a homeomorphism such that there exists a subcontinuum $C \subseteq P$, $x \neq y \in C$ with the property that*

(1) $\sigma \restriction C = \mathrm{id}_C$, and
(2) whenever U_1, U_2, U_3 are pairwise disjoint regular open sets in P with
   (a) $\bigcup_{i=1}^{3} \mathrm{cl}(U_i) = P$,
   (b) $\bigcap_{i=1}^{3} \mathrm{cl}(U_i) = \emptyset$ (that is, each point has a neighborhood which intersects only two of the U_i's),
   (c) $C \subseteq \mathrm{int}(\mathrm{cl}(V_2) \cup \mathrm{cl}(V_3))$ (equivalently, $\mathrm{cl}(U_1) \cap C = \emptyset$, C is positive distance from U_1),
   (d) $x \in U_2$, $y \in U_3$,
   then $\sigma(U_2) \cap U_3 \neq \emptyset$, or $\sigma(U_3) \cap U_2 \neq \emptyset$.

Then σ is not conjugate to any member of Aut$(\mathbb{P})$.

Proof. Note that the above property of σ is invariant under conjugation by a homeomorphism, therefore it suffices to show that representing P as a quotient of the prespace $\mathbb{P}$, necessarily $\sigma \notin \mathrm{Aut}(\mathbb{P})$.

So we can fix a generic sequence I_j, $\pi_{i,j}$ $(i \leq j \in \mathbb{N})$ so that $\mathbb{P} = \operatorname{proj\,lim}_j I_j$, then $P \approx \mathbb{P}/R^{\mathbb{P}}$. We also fix the canonical surjection $\mathrm{pr} : \mathbb{P} \to P$.

Suppose that σ is an automorphism, which formally means that there exists some $\tilde{\sigma} : \mathbb{P} \to \mathbb{P}$ with $\mathrm{pr}(\tilde{\sigma}) = \sigma$ (where $\mathrm{pr} : \mathrm{Aut}\mathbb{P} \to \mathrm{Homeo}(P)$ is the canonical embedding; in other words, the equality $\sigma \circ \mathrm{pr} = \mathrm{pr} \circ \tilde{\sigma}$ holds).

We use that $J_j := \pi_j \circ \mathrm{pr}^{-1}(C)$ is a nonempty subinterval in I_j, and for large enough j we have $|J_j| > 2$ (otherwise its image under pr is a singleton). In other words, $\mathrm{pr}^{-1}(C) = \operatorname{proj\,lim}_j J_j$.

For each j we let $X_j = \pi_j \circ \mathrm{pr}^{-1}(x)$, $Y_j = \pi_j \circ \mathrm{pr}^{-1}(y)$, so $X_j, Y_j \subseteq I_j$, $1 \leq |X_j|, |Y_j| \leq 2$. Let j_0 be large enough so that $X_{j_0} \cup Y_{j_0}$ spans a not connected subgraph in $J_{j_0} \subseteq I_{j_0}$ (i.e. there is a gap in between), moreover, each connected component of $I_k \setminus (X_k \cup Y_k)$ has at least two elements.

Pick a large enough k so that $\pi_{j_0} \circ \tilde{\sigma}$ factors through π_k, i.e. for some epimorphism $\alpha : I_k \to I_{j_0}$ we have

$$\pi_{j_0} \circ \tilde{\sigma} = \alpha \circ \pi_k. \tag{5.1}$$

Claim 5.3. $\alpha \restriction J_k = \pi_{j_0,k}$.

First we argue that the claim will finish the proof of the lemma. Pick disjoint subsets $X', Y' \subseteq J_{j_0}$ with $X' \cup Y' = J_{j_0}$, $X_{j_0} \subseteq X'$ and $Y_{j_0} \subseteq Y'$. Let $X^+ = \pi_{j_0,k}^{-1}(X') \cap J_k$, $Y^+ = \pi_{j_0,k}^{-1}(Y') \cap J_k$, in particular, $X^+ \cup Y^+ = J_k$, and

$$\pi_{j_0,k}[X^+] \cap \pi_{j_0,k}[Y^+] = \emptyset, \tag{5.2}$$

$$X_k \subseteq X^+, \quad Y_k \subseteq Y^+, \tag{5.3}$$

Let $V_1 = I_k \setminus J_k$ be the union of the two complementer intervals. Now $W_1 = \mathrm{pr}(\pi_k^{-1}(V_1)), W_2 = \mathrm{pr}(\pi_k^{-1}(X^+)), W_3 = \mathrm{pr}(\pi_k^{-1}(Y^+))$ are closed sets that cover P. The fact that $X^+ \cup Y^+ = J_k = \pi_k[\mathrm{pr}^{-1}(C)]$ together with $V_1 \cap J_k = \emptyset$ imply $C \cap W_1 = \emptyset$. Also, it follows from (5.3) and the way X_k, Y_k were defined that

$$x \notin W_1 \cup W_3,$$
$$y \notin W_1 \cup W_2.$$

Recalling Lemma 2.1 and using that $\mathcal{K} = \mathcal{P}$ is the class of finite linear graphs it is routine to check that for any $S \subseteq I_k$, letting $D_S \subseteq \mathbb{P}$ denote the points that project to S (i.e. $D_S = \pi_k^{-1}(S)$),

$$\mathrm{int}(\mathrm{pr}(D_S)) = \mathrm{pr}(D_S) \setminus \mathrm{pr}(D_{I_k \setminus S}),$$

and $\mathrm{cl}(\mathrm{int}(\mathrm{pr}(D_S))) = \mathrm{pr}(D_S))$, so $U_i = \mathrm{int}(W_i)$, $i = 1,2,3$ are pairwise disjoint regular open sets satisfying the requirements above.

We assume that $\sigma(U_2) \cap U_3 \neq \emptyset$ as the other case can be dealt with the same way. So pick $z \in \mathbb{P}$ witnessing this, i.e. $\mathrm{pr}(z) \in U_2$, $\sigma(\mathrm{pr}(z)) = \mathrm{pr}(\tilde{\sigma}(z)) \in U_3$. By the definition of U_2, we have $\mathrm{pr}(z) \in W_2 \setminus (W_1 \cup W_3)$, so $\pi_k(z) \in X^+$ necessarily. Similarly, $\pi_k(\tilde{\sigma}(z)) \in Y^+$, so

$$\pi_{j_0}(\tilde{\sigma}(z)) = \pi_{j_0,k}(\pi_k(\tilde{\sigma}(z))) \in \pi_{j_0,k}[Y^+]. \tag{5.4}$$

But by (5.1) $\pi_{j_0}(\tilde{\sigma}(z)) = \alpha(\pi_k(z))$, so by Claim 5.3

$$\pi_{j_0}(\tilde{\sigma}(z)) = \alpha(\pi_k(z)) = \pi_{j_0,k}(\pi_k(z)) \in \pi_{j_0,k}[X^+]. \tag{5.5}$$

Finally, note that (5.4), (5.5) contradict (5.2).

Proof. (Claim 5.3) If $u \in J_k$, $v' = \alpha(u) \neq \pi_{j_0,k}(u)$, then letting u' denote $\pi_{j_0,k}(u)$, clearly $\pi_{j_0,k}^{-1}(u') \cap \pi_{j_0,k}^{-1}(v') = \emptyset$. We now claim the following.

($\circledast$) There exists $k^+ \geq k$, such that one of the following two possibilities holds:
- for some $u^+ \in J_{k^+}$ we have $\pi_{k,k^+}(u^+) = u$, but $\neg(u^+ R \pi_{j_0,k^+}^{-1}(v'))$,
- for some $v^+ \in J_{k^+}$ with $v = \pi_{k,k^+}(v^+)$ we have $\pi_{j_0,k}(v) = v'$, $\alpha(v) = u'$ and $\neg(v^+ R \pi_{j_0,k}^{-1}(u'))$.

Before verifying ($\circledast$) we argue why this is enough to finish proving Claim 5.3. Clause ($\circledast$) asserts that possibly replacing k with a bigger number k^+, and u with an element of $\pi_{k,\ell}^{-1}(u)$ (and possibly interchanging the roles of the u's and v's) we can assume that

$$\neg(u R \pi_{j_0,k}^{-1}(v')), \tag{5.6}$$

i.e. there is no adjacent node to u in $\pi_{j_0,k}^{-1}(v')$. This would yield a contradiction, as picking any thread $(u_i)_i$ in $\operatorname{proj\,lim}_j J_j$ with $u_k = u$, then $(u_i)_i \in \mathrm{pr}^{-1}(C)$, but for the thread $(v_i)_{i\in\omega}$ defined by the equation $(v_i)_i = \tilde{\sigma}((u_i)_i) \in \mathbb{P}$ necessarily $v_{j_0} = \alpha(u) = v'$, so $v_k \in \pi^{-1}_{j_0,k}(v')$, so $\neg(v_k R u_k)$. This means that $\neg(\tilde{\sigma}((u_i)_i)R((u_i)_i))$, and so $\sigma(\mathrm{pr}((u_i)_i) \neq \mathrm{pr}((u_i)_i)$, contradicting our premise $\sigma\restriction C = \mathrm{id}_C$. Therefore it remains to verify ($\circledast$).

We assume that the first alternative fails. Then a standard argument shows that for some $(u_i)_i$ with $u_{j_0} = u'$, and $(v_i)_i$ with $v_{j_0} = v'$ we have $\tilde{\sigma}((u_i)_i) = (v_i)_i$ and $u_i R v_i$ for all i. Then (using that every node is of degree at most 1 in the graph $R^{\mathbb{P}}$), $\tilde{\sigma}((v_i)_i) = (u_i)_i$, in particular, $\alpha(v_k) = u_{j_0}$.

Now if both alternative fails, then for each $k^+ \geq k$ we have that both $J_{k^+} \cap \pi^{-1}_{k,k^+}(u)$ and $J_{k^+} \cap \pi^{-1}_{j_0,k^+}(v')$ are the union of intervals each of length at most 2. But by Lemma 2.1 we know that for some large enough k^+ both $\pi^{-1}_{k,k^+}(u)$ and $\pi^{-1}_{j_0,k^+}(v')$ are unions of intervals with each of length at least 3, therefore $J_{k^+} \cap \pi^{-1}_{k,k^+}(u)$ and $J_{k^+} \cap \pi^{-1}_{j_0,k^+}(v')$ are two intervals with each one being adjacent to $I_{k^+} \setminus J_{k^+}$. Since they are adjacent to each other (by our assumptions), we obtain that $|J_{k^+}| \leq 4$, which would imply that $|\operatorname{proj\,lim}_j J_j| \leq 4$. Finally, note that $|(\operatorname{proj\,lim}_j J_j)/R| = 1$, which contradicts $x \neq y$, as both lie in $\mathrm{pr}(\operatorname{proj\,lim}_j J_j)$. This finishes the proof of ($\circledast$). $\square_{\text{Claim5.3}}$

$\square_{\text{Lemma5.2}}$

The only thing en route to a homeomorphism that is not conjugate to any element of $\mathrm{Aut}(\mathbb{P})$ is to construct one that satisfies the premise of Lemma 5.2.

Proposition 5.4. *There exists an $\sigma \in \mathrm{Homeo}(P)$ as in Lemma 5.2.*

Proof. We are going to

- define a generic sequence $J_0, J_1, \dots, J_n, \dots$ of finite linear graphs by hand (together with the bonding maps $\pi_{i,j}$, $i \leq j <$),
- identify $\mathbb{P}$ with $\operatorname{proj\,lim}_i J_i$, and let $\pi_j : \operatorname{proj\,lim}_i J_i : J_j$ be the natural projection,
- and construct epimorphisms $h_i : J_{n_i+1} \to J_{n_i}$ in such a way that for any choice of $\tilde{\sigma}_i \in \mathrm{Aut}(\mathbb{P})$ with $\pi_{n_i} \circ \tilde{\sigma}_i = h_i \circ \pi_{n_i+1}$ we have that $\tilde{\sigma}_i$ converges uniformly (in $\mathcal{C}(P,P)$), moreover the limit is an element of $\mathrm{Homeo}(P)$,

and we will let σ be the obtained function $\lim_i \tilde{\sigma}_i$.

The following definitions are particular cases of notions from an unpublished work of Solecki and Tsankov.

Definition 5.5 (Solecki–Tsankov)**.** Let L be a finite linear graph. A family t of sets is an *L-type* if t is a maximal linearly ordered by $\subseteq$ family of connected subsets of L.

Definition 5.6 (Solecki–Tsankov)**.** Suppose that $f : J \to L$ is an epimorphism between the finite linear graphs J, L, and let $a \in J$ be a node, $M \subseteq J$ be a subinterval with a being an endpoint of M. Then we define the type of the pair a, L with respect to f, in symbols, $\mathrm{tp}^{a,M}(f)$ to be the L-type

$$\mathrm{tp}^{a,M}(f) := \{f[M'] : \ M' \subseteq M \text{ is an interval, } a \in M'\}.$$

(So if $M = \{a, a', a'', \dots, a^{(|M|-1)}\}$ is an enumeration of M that is a walk, then $tp^{a,M}(f)$ codes the order in which the walk $f(a), f(a'), \dots f(a^{(|M|-1)})$ visits the nodes of J.)

By induction on i we define

- J_i, K_i, $\pi_{i,i+1}$ for $i = -1, 0, 1, \dots$,
- h_{-1},
- $h^\bullet_k$, $k \in \omega$,
- L_{4k+2}, L_{4k+3} $(k \in \omega)$,
- g_k, f_k, f'_k, h_k $(k \in \omega)$,

keeping the following outline in mind. The idea is that $\operatorname{proj\,lim}_i K_i \subseteq \operatorname{proj\,lim}_i J_i$ will represent the subcontinuum C on which the prospective homeomorphism σ is the identity. σ will be approximated with automorphisms represented by the h_i's, or f_i's, g_i's. Basically these are liftings of each other, the only purpose to give them different names is an attempt to ease the notational awkwardness later, e.g. in the proof of Lemma 5.7. More concretely, $h_i : J_{4i} \to J_{4i-1}$ is an epimorphism and $h^\bullet_i : J_{4i+1} \to J_{4i}$ is an epimorphism with $h_i \circ h^\bullet_i = \pi_{4i-1,4i+1}$, where the existence of $h^\bullet_i$'s will ensure that the limit is (left-)invertible, so injective. Then $f_i : J_{4k+2,4k+1}$ will be a lifting of h_i, which will be ensured by the condition $h^\bullet_i \circ g_i = \pi_{4i,4i+2}$. In the recursive construction K_{4i+1} will not only have K_{4i+2} as a g_i and $\pi_{4i+1,4i+2}$-preimage (on which g_i and $\pi_{4i+1,4i+2}$ coincide), but also a copy of it L_{4i+2}. In the next step of the recursion this is followed by the construction of J_{4i+3}, and $f_i, \pi_{4i+2,4i+3} : J_{4i+3} \to J_{4i+2}$, where we can also guarantee that $f_i, \pi_{4i+2,4i+3}$ map L_{4i+3} to L_{4i+2}, as well as K_{4i+2} to K_{4i+2}. But instead of f_i being a lifting of g_i, it will be only an almost lifting, because we will arrange so that $f_i \restriction L_{4i+3}$ is roughly speaking a shift of $\pi_{4i+3,4i+2} \restriction L_{4i+3}$ by one. Finally, $h_{i+1} : J_{4i} \to J_{4i.1}$ will be a lifting of f_i guaranteed by the condition $f_i \circ \pi_{4i+3,4i+4} = \pi_{4i+2,4i+3} \circ h_{i+1}$. Each h_i will be responsible an amalgamation task. We will have further technical conditions, e.g. the conditions on types, which will be necessary to carry out our main tasks.

Formally we require that the J_i, K_i, f_i, f'_i, g_i, h_i, $h^\bullet_i$ satisfy the following.

- $\blacksquare_1$ J_i is a finite linear graph,
- $\blacksquare_2$ $K_i \subseteq J_i$ is a subinterval,
- $\blacksquare_3$ $K_{-1} = L_{-1} = J_{-1}$ are the one element linear graph,
- $\blacksquare_4$ h_i, $h^\bullet_i$, g_i, f_i, f'_i are epimorphisms
  - $h_i \in \operatorname{Epi}(J_{4i}, J_{4i-1})$,
  - $h^\bullet_i \in \operatorname{Epi}(J_{4i+1}, J_{4i})$,
  - $g_i \in \operatorname{Epi}(J_{4i+2}, J_{4i+1})$,
  - $f_i, f'_i \in \operatorname{Epi}(J_{4i+3}, J_{4i+2})$,
- $\blacksquare_5$ $\pi_{i,i+1}[K_{i+1}] = K_i$, and the π's agree with the $h_i/h^\bullet_i/g_i/f_i$ on K_k whenever it is appropriate (and defined), i.e.
  - $\pi_{4k-1,4k} \restriction K_{4k} = h_k \restriction K_{4k}$,
  - $\pi_{4k,4k+1} \restriction K_{4k+1} = h^\bullet_k \restriction K_{4k+1}$,
  - $\pi_{4k+1,4k+2} \restriction K_{4k+2} = g_k \restriction K_{4k+2}$,
  - $\pi_{4k+2,4k+3} \restriction K_{4k+3} = f_k \restriction K_{4k+3}$,

  moreover they map endpoints to endpoints,

$\blacksquare_6$ for each $a, b \in J_{i+1}$, if aR^3b, then $\pi_{i,i+1}(a)R\pi_{i,i+1}(b)$ (e.g. if for $c \in J_i$, each connected component in $\pi^{-1}_{i,i+1}(c)$ is an interval of length at least 3),

$\blacksquare_7$ for $i = 4k - 1$ (including $i = -1$):

$\blacksquare_7(a)$ for each maximal connected component (subinterval) C of $J_{4k} \setminus K_{4k}$ we have $\pi_{4k-1,4k}[C] = J_{4k-1} = h_k[C]$, and

$$\mathrm{tp}^{c^*,C}(h_k) = \mathrm{tp}^{c^*,C}(\pi_{4k-1,4k}) \text{ for } c^* \in C \text{ with } c^*RK_{4k},$$

$\blacksquare_7(b)$ (and if $k \geq 0$:) $h_{4k-2} \circ \pi_{4k-1,4k} = \pi_{4k-2,4k-1} \circ h_{4k-1}$,

$\blacksquare_8$ for $i = 4k$:

$\blacksquare_8(a)$ $h_k \circ h^\bullet_k = \pi_{4k-1,4k} \circ \pi_{4k,4k+1}$,

$\blacksquare_8(b)$ for each maximal connected component (subinterval) C of $J_{4k+1} \setminus K_{4k+1}$ we have $\pi_{4k,4k+1}[C] = J_{4k} = h_k[C]$, and

$$\mathrm{tp}^{x,C}(h_k) = \mathrm{tp}^{x,C}(\pi_{4k,4k+1}) \text{ for } x \in C \text{ with } xR^{J_{4k+1}}K_{4k+1},$$

$\blacksquare_8(c)$ for each $a, b \in J_{4k+1}$, if aR^3b, then $h^\bullet_k(a)Rh^\bullet_k(b)$,

$\blacksquare_9$ for $i = 4k + 1$:

$\blacksquare_9(a)$ $\pi_{4k+1,4k+2}[L_{4k+2}] = K_{4k+1}$, and $\pi_{4k+1,4k+2}\restriction L_{4k+2}$ maps endpoints to endpoints,

$\blacksquare_9(b)$ $\pi_{4k+1,4k+2}\restriction L_{4k+2} = g_k\restriction L_{4k+2}$,

$\blacksquare_9(c)$ $\neg(L_{4k+2}RK_{4k+2})$,

$\blacksquare_9(d)$ for each maximal connected component (subinterval, of which there are 3) C of $J_{4k+2} \setminus (K_{4k+2} \cup L_{4k+2})$ we have

$$\pi_{4k+1,4k+2}[C] = J_{4k+1} = g_k[C],$$

and

$$\begin{gathered}\text{for } x \in C \text{ with } xR^{J_{4k+2}}(K_{4k+2} \cup L_{4k+2}) \\ \mathrm{tp}^{x,C}(g_k) = \mathrm{tp}^{x,C}(\pi_{4k+1,4k+2}),\end{gathered}$$

$\blacksquare_9(e)$ $h^\bullet_k \circ g_k = \pi_{4k,4k+1} \circ \pi_{4k+1,4k+2}$,

$\blacksquare_{10}$ for $i = 4k + 2$:

$\blacksquare_{10}(a)$ $\pi_{4k+2,4k+3}[L_{4k+3}] = L_{4k+2}$, $\pi_{4k+2,4k+3}\restriction L_{4k+3}$ maps endpoints to endpoints,

$\blacksquare_{10}(b)$ $\pi_{4k+2,4k+3}\restriction L_{4k+3} = f'_k\restriction L_{4k+3}$,

$\blacksquare_{10}(c)$ letting C_1, C_2, C_3 denote the connected components in $J_{4k+2} \setminus (K_{4k+2} \cup L_{4k+2})$ we have that letting

$$C'_i := \pi^{-1}_{4k+2,4k+3}(C_i), \; i = 1, 2, 3,$$

C'_1, C'_2, C'_3 are all connected (intervals), and

$$C'_j = (f'_k)^{-1}(C_j) \text{ for } j = 1, 2, 3.$$

$\blacksquare_{10}(d)$ $\pi_{4k+1,4k+2} \circ f'_k = g_k \circ \pi_{4k+2,4k+3}$,

$\blacksquare_{10}(e)$ for $a \in J_{4k+3} \setminus L_{4k+3}$ we have $f_k(a) = f'_k(a)$,

$\blacksquare_{10}(f)$ for each $a \in L_{4k+3}$ we have $|f_k(a) - f'_k(a)| \leq 2$, and if $a \in L_{4k+3}$ is such that $\pi_{4k+2,4k+3}(a) \in L_{4k+2}$ is neither an endpoint nor is related to an endpoint of L_{4k+2}, then

$$|f_k(a) - f'_k(a)| = 2 \; (= |\pi_{4k+2,4k+3}(a) - f'_k(a)|)$$

(note that the above imply

$$(\pi_{4k+1,4k+2} \circ f_k) R (g_k \circ \pi_{4k+2,4k+3}), \tag{5.7}$$

- $\blacksquare_{11}$ whenever M is a linear graph and $\phi : M \to J_i$ for some i, then there exists $j \geq i$, $\phi^+ : J_j \to M$, such that $\pi_{i,j} = \phi \circ \phi^+$ (where $\pi_{i,j}$ denotes the composition $\pi_{i,i+1} \circ \ldots \circ \pi_{j-1,j}$),

Before the construction we argue that this will result the required homeomorphism.

Lemma 5.7. *Assume that the induction maintaining* $\blacksquare_1$- $\blacksquare_{11}$ *can be carried out. Then*

(1) $\operatorname{proj\,lim}_i J_i$ *(with the bonding maps* $\pi_{i,j}$, $i \leq j$ *defined as in* $\blacksquare_{11}$*) is isomorphic to* $\mathbb{P}$,
(2) *letting* $K = \operatorname{proj\,lim}_i K_i (\subseteq \operatorname{proj\,lim}_i J_i)$, *its projection* $C := \operatorname{pr}[K]$ *is a subcontinuum,*
(3) *there exists* $(x_i)_i$, $(y_i)_i \in K$ *such that* $\pi_i((x_i)_i) = x_i$, $\pi_i((y_i)_i) = y_i$ *are the two endpoints of* K_j,
(4) *if for each* k *the map* $\sigma_k \in \operatorname{Aut}(\mathbb{P})$ *is such that* $\pi_{4k-1} \circ \sigma_k = h_k \circ \pi_{4k}$, *then* σ_k *is convergent in* $\operatorname{Homeo}(P)$, *and their limit* σ *is as in Lemma 5.2 witnessed by the subcontinuum* C *and* $x = \operatorname{pr}((x_i)_i)$, $y = \operatorname{pr}((y_i)_i)$.

Proof. (Lemma 5.7) It is standard that $\blacksquare_{11}$ suffices for (1).

Since K_i is an interval, any clopen decomposition $C_0 \cup C_1$ of K is of the form $C_0 = \pi_i^{-1}(K_i^0)$, $C_1 = \pi_i^{-1}(K_i^1)$, for a partition $K_i^0 \cup K_i^1$ of K_i. Now there must exist $(z_i^0)_i, (z_i^1)_i \in K$ with $z_i^0 \in K_i^0$, $z_i^1 \in K_i^1$ and $(z_i^0)_i R (z_i^1)_i$. But then necessarily $\operatorname{pr}((z_i^0)_i) = \operatorname{pr}((z_i^1)_i)$, so C is a continuum, indeed.

For (3) note that $\{x_i, y_i\}$ must be the set of the two endpoints of K_i. Now clause $\blacksquare_5$ clearly implies (3).

Pick such an σ_k for each k. First we are going to check that $(\sigma_k)_k$ is convergent in $\mathcal{C}(P,P)$ (so $\lim_k \sigma_k$ is a surjective continuous function), and that $\lim_k \sigma_k$ is injective. In light of the second part of Theorem 2.4 it suffices to verify that for all finite connected linear graph A and $\varphi \in \operatorname{Epi}(\mathbb{P}, A)$ we have that for all large enough n, n' we have $(\varphi \circ \sigma_n) R (\varphi \circ \sigma_{n'})$. Using the fact that every epimorphism factors through a π_j, it is enough to check that

$$(\pi_{4k-1} \circ \sigma_k) R (\pi_{4k'-1} \circ \sigma_{k'}) \quad \text{if } k \leq k'. \tag{5.8}$$

A standard induction argument gives that the following claim implies (5.8).

Claim 5.8. *If* $\psi \in \operatorname{Epi}(\mathbb{P}, J_{4k+3})$ *is such that* $\psi R (\pi_{4k+3} \circ \sigma_{k+1})$, *then*

$$(\pi_{4k-1,4k+3} \circ \psi) R (\pi_{4k-1} \circ \sigma_k).$$

Proof. (Claim 5.8) First we note that since $\pi_{4k+3} \circ \sigma_{k+1} = h_{k+1} \circ \pi_{4k+4}$, clause $\blacksquare_7(b)$ implies

$$\begin{aligned} \pi_{4k+2} \circ \sigma_{k+1} = \pi_{4k+2,4k+3} \circ h_{k+1} \circ \pi_{4k+4} = f_k \circ \pi_{4k+3,4k+4} \circ \pi_{4k+4} \\ = f_k \circ \pi_{4k+3}. \end{aligned} \tag{5.9}$$

As (5.7) says

$$(\pi_{4k+1,4k+2} \circ f_k) R (g_k \circ \pi_{4k+2,4k+3}),$$

we can apply $\pi_{4k+1,4k+2}$ to (5.9)

$$\pi_{4k+1} \circ \sigma_{k+1} = \pi_{4k,4k+2} \circ f_k \circ \pi_{4k+3} \ R \ (g_k \circ \pi_{4k+2}). \tag{5.10}$$

On the other hand, applying $\pi_{4k+1,4k+3}$ to $\psi R(\pi_{4k+3} \circ \sigma_{k+1})$ we get

$$(\pi_{4k+1,4k+3} \circ \psi)R(\pi_{4k+1} \circ \sigma_{k+1}),$$

so by (5.10),

$$(g_k \circ \pi_{4k+2})R^2(\pi_{4+1,4k+3} \circ \psi).$$

By $\blacksquare_8(a)$ we can apply $h_k \circ h_k^\bullet = \pi_{4k-1,4k} \circ \pi_{4k,4k+1} = \pi_{4k-1,4k+1}$ to both sides, to get

$$(h_k \circ h_k^\bullet \circ g_k \circ \pi_{4k+2})R(\pi_{4k-1,4k+3} \circ \psi)$$

(here we can write R instead of R^2 by $\blacksquare_6$). Now using $\blacksquare_9(e)$, the LHS can be simplified to $(h_k \circ \pi_{4k})$, so

$$(h_k \circ \pi_{4k})R(\pi_{4k-1,4+3} \circ \psi),$$

but notice that this LHS is just $\pi_{4k-1} \circ \sigma_k$, and we are done. $\square_{\text{Claim5.8}}$

This means that (5.8) holds, and $\sigma := \lim_k \sigma_k$ exists (in $\mathcal{C}(P, P)$). We now check that σ is a homeomorphism, for which it is enough to see that σ is injective.

Claim 5.9. *The map $\sigma = \lim_k \sigma_k \in \mathcal{C}(P, P)$ is injective.*

Proof. (Claim 5.9) Pick $(z_i)_i$, $(z_i')_i$ in $\mathbb{P}$ with $\mathrm{pr}((z_i)_i) \neq \mathrm{pr}((z_i')_i)$. This means that $\neg(z_i R z_i')$ for large enough i. With (5.8) it is enough to show that for large enough k we have $\neg(\pi_{4k-1} \circ \sigma_k((z_i)_i)R^3\pi_{4k-1} \circ \sigma_k((z_i')_i)$, i.e (by the definition of σ_k), we need that

$$\neg(h_k(z_{4k})R^3h_k(z_{4k}')).$$

Suppose that k is large enough so that

$$\neg(z_{4k}Rz_{4k}'). \tag{5.11}$$

We claim that $\neg(h_{k+1}(z_{4k+4})R^3h_{k+1}(z_{4k+4}'))$. Assume otherwise, so

$$h_{k+1}(z_{4k+4})R^3h_{k+1}(z_{4k+4}'),$$

and therefore

$$(\pi_{4k+2,4k+3} \circ h_{k+1}(z_{4k+4}))R(\pi_{4k+2,4k+3} \circ h_{k+1}(z_{4k+4}') \tag{5.12}$$

by $\blacksquare_6$. Applying $\blacksquare_7(b)$ to the LHS,

$$(\pi_{4k+2,4k+3} \circ h_{k+1}(z_{4k+4})) = (f_k \circ \pi_{4k+3,4k+4})(z_{4k+4}) = f_k(z_{4k+3}), \tag{5.13}$$

and similarly for z_{4k+4}', i.e.

$$(\pi_{4k+2,4k+3} \circ h_{k+1}(z_{4k+4}')) = (f_k \circ \pi_{4k+3,4k+4})(z_{4k+4}') = f_k(z_{4k+3}'). \tag{5.14}$$

Combining (5.12)-(5.14) we get $(f_k(z_{4k+3})R(f_k(z_{4k+3}')$, so clearly

$$(\pi_{4k+1,4k+2} \circ f_k(z_{4k+3})R(\pi_{4k+1,4k+2} \circ f_k(z_{4k+3}'). \tag{5.15}$$

Applying (5.7),

$$(\pi_{4k+1,4k+2} \circ f_k(z_{4k+3}))R(h_{4k+1} \circ \pi_{4k+2,4k+3})(z_{4k+3}) = g_k(z_{4k+2}), \tag{5.16}$$

and similarly for z', i.e.

$$(\pi_{4k+1,4k+2} \circ f_k(z_{4k+3}'))R(h_{4k+1} \circ \pi_{4k+2,4k+3})(z_{4k+3}') = g_k(z_{4k+2}'), \tag{5.17}$$

and so (5.15)-(5.17) give us $g_k(z_{4k+2})R^3 g_k(z'_{4k+2})$. Then by $\blacksquare_8(c)$

$$(h^\bullet_k \circ g_k(z_{4k+2}))R(h^\bullet_k \circ g_k(z'_{4k+2})).$$

By ($\blacksquare_9(e)$), $h^\bullet_k \circ g_k(z_{4k+2}) = z_{4k}$, and $h^\bullet_k \circ g_k(z'_{4k+2}) = z'_{4k}$, so $z_{4k}Rz'_{4k}$, which contradicts (5.11). $\square_{\text{Claim5.9}}$

So $\sigma \in \mathrm{Homeo}(P)$, indeed, and it remains to check that C, x, y witness that σ has the properties from Lemma 5.2. First we recall Fact A.2, i.e. that sets of the form

$$\{\mathrm{pr}((z_i)_i) : \ (z_i)_i \in \mathbb{P}, \ z_j R x_j\} \ (j \in \mathbb{N})$$

form a neighborhood basis of $x = \mathrm{pr}((x_i)_i)$, and similarly with $y = \mathrm{pr}((y_i)_i)$.

Now we can fix the pairwise disjoint regular open sets U_1, U_2, U_3 of P as in Lemma 5.2 with $x \in U_2$, $y \in U_3$, $\bigcup_{i=1}^4 \mathrm{cl}(U_i) = P$ and $C \cap \mathrm{cl}(U_1) = \emptyset$. Let $W_i = \mathrm{pr}^{-1}(U_i)$ for $i = 1,2,3$. By the observation above there exists i_0 such that

$\boxdot_1$ for every $(z_i)_i \in \mathbb{P}$, $z_{i_0} R x_{i_0}$ implies $\mathrm{pr}((z_i)_i) \in U_2$, so $(z_i)_i \in W_2$.

Similarly,

$\boxdot_2$ $z_{i_0} R y_{i_0}$ implies that $(z_i)_i \in W_3$.

Since each point admits a neighborhood which at most two U_i can intersect, we can also assume that i_0 is large enough so that for $a \in J_{i_0}$

$$\pi^{-1}_{i_0}(a) \text{ intersects at most 2 of } \{W_1, W_2, W_3\}.$$

W.l.o.g. we can assume that $i_0 = 4k$ for some $k \in \mathbb{N}$.

Since C is of positive distance from U_1, we may also assume that whenever $a \in K_{4k}$ we have $\pi^{-1}_{4k}(a)$ intersects only $W_2 \cup W_3$, but cannot intersect W_1. Moreover, since $\mathrm{pr}(\pi^{-1}_{4k}(a))$ has nonempty interior in P and $\bigcup_{i=1}^3 U_i$ is dense, at least one of W_2 and W_3 must intersect $\pi^{-1}_{4k}(a)$. This means that

$\boxdot_3$ if $a \in L_{4k+2}$, then $\pi^{-1}_{4k+2}(a)$ can only intersect W_2 and W_3, and it intersects at least one of them.

Moreover, if $L_{4k+2} = \{l_j : \ j < |L_{4k+2}|\}$ (with $l_j R l_{j+1}$), then

$$\{\pi_{4k,4k+2}(l_0), \pi_{4k,4k+2}(l_{|L_{4k+2}|-1})\} = \{x_{4k}, y_{4k}\}$$

by $\blacksquare_5$, $\blacksquare_9(a)$. Therefore, it follows from $\boxdot_1$, $\boxdot_2$ together with $\blacksquare_6$ that

$\boxdot_4$ $\pi^{-1}_{4k+2}(l_0) \cup \pi^{-1}_{4k+2}(l_1) \cup \pi^{-1}_{4k+2}(l_2) \subseteq W_2$,

$\boxdot_5$ $\pi^{-1}_{4k+2}(l_{|L_{4k+2}|-1}) \cup \pi^{-1}_{4k+2}(l_{|L_{4k+2}|-2}) \cup \pi^{-1}_{4k+2}(l_{|L_{4k+2}|-3}) \subseteq W_3$

which we can assume by possibly flipping the order and numbering (and in fact this is not even strict, but we won't need more than these 3-3 elements). Now we recall $\blacksquare_{10}(f)$, i.e. for $a \in L_{4k+3}$ we have $|f_k(a) - \pi_{4k+2,4k+3}(a)| = 2$ unless $\pi_{4k+2,4k+3}(a) \in \{l_0, l_1, l_{|L_{4k+2}|-1}, l_{|L_{4k+2}|-2}\}$. This can only happen if either

$\boxdot_6$ for every a with $\pi_{4k+2,4k+3}(a) \notin \{l_0, l_1, l_{|L_{4k+2}|-1}, l_{|L_{4k+2}|-2}\}$ we have

$$f_k(a) = l_{\pi_{4k+2,4k+3}(a)+2},$$

or for almost all a's $f_k(a) = l_{\pi_{4k+2,4k+3}(a)-2}$ holds. We will assume $\boxdot_6$, as the other case is essentially the same. We

$\boxdot_7$ let $m < |L_{4k+2}|$ be largest such that $\pi^{-1}_{4k+2}(l_m) \cap W_2 \neq \emptyset$, and note that

$$2 \leq m < |L_{4k+2}| - 3 \tag{5.18}$$

holds by $\boxdot_5$ (since the W_j's are pairwise disjoint). Pick $\bar{z} = (z_i)_i \in W_2 \cap \pi^{-1}_{4k+2}(l_m)$. Now $\pi_{4k+3,4k+2}(z_{4k+3}) = z_{4k+2} = l_m$, so necessarily

$$f_k(z_{4k+3}) = l_{m+2}. \tag{5.19}$$

Recalling (5.8) and the fact that $\sigma(\mathrm{pr}(\bar{z}))$ is the limit of $\mathrm{pr}(\sigma_i(\bar{z}))$ (in P) by compactness of $\mathbb{P}$ there exists a sequence $\bar{z}^* = (z^*_i)_i \in \mathbb{P}$ that is a an accumulation point of $(\sigma_i(\bar{z}))_i$, so necessarily represents $\sigma(\mathrm{pr}(\bar{z})) \in P$, i.e. $\mathrm{pr}(\bar{z}) = \sigma(\mathrm{pr}(\bar{z}))$ and has the property

$$z^*_{4k+3}\ R\ (\pi_{4k+3} \circ \sigma_{k+1}(\bar{z})) = h_{k+1}(z_{4k+4})$$

(where the equality follows from the way we picked σ_k, (4)). This means that the following claim implies $(z^*_i)_i \in W_3$, so $\mathrm{pr}((z^*_i)_i) \in U_3$, and we are done.

Claim 5.10. *If $b \in J_{4k+3}$ is such that $bRh_{k+1}(z_{4k+4})$, then $\pi^{-1}_{4k+3}(b) \subseteq W_3$.*

Proof. First we let $w = h_{k+1}(z_{4k+4}) \in L_{4k+3}$, note that (by ($\blacksquare_7(b)$))

$$\pi_{4k+2,4k+3}(w) = \pi_{4k+2,4k+3}(h_{k+1}(z_{4k+4})) = f_k(z_{4k+3}) = l_{m+2},$$

in particular, $\neg(\pi_{4k+2,4k+3}(w)Rl_m$, so by $\blacksquare_6$ and (5.18)

$$wR^3b \ \Rightarrow\ \pi_{4k+2,4k+3}(b) \in \{l_{m+1}, l_{m+2}, l_{m+3}\} \subseteq L_{4k+2}. \tag{5.20}$$

This means bR^3w implies $\pi^{-1}_{4k+3}(b) \cap W_2 = \emptyset$ by the definition of m ($\boxdot_7$) Hence, by the definition of the W_j's (i.e. $W_i = \mathrm{pr}^{-1}(U_i)$)

$$\mathrm{pr}(\pi^{-1}_{4k+3}(b)) \cap U_2 = \emptyset \text{ if } bR^3w, \tag{5.21}$$

similarly, by $\boxdot_3$

$$\mathrm{pr}(\pi^{-1}_{4k+3}(b)) \cap U_1 = \emptyset \text{ if } bR^3w. \tag{5.22}$$

To summarize, we got

$$\bigcup_{b \in J_{4k+3},\ bR^3w} \mathrm{pr}(\pi^{-1}_{4k+3}(b)) \cap (U_1 \cup U_2) = \emptyset.$$

Since $\bigcup_{j=1}^3 U_j$ is dense in P, U_3 must be dense in the open set

$$P \setminus \left(\bigcup_{b \in J_{4k+3},\ \neg(bR^3w)} \mathrm{pr}(\pi^{-1}_{4k+3}(b)) \right),$$

so using that $\mathrm{pr}(\pi^{-1}_{4k+3}(d))$, $\mathrm{pr}(\pi^{-1}_{4k+3}(c))$ are closed sets that are from positive distance apart iff $\neg(cRd)$,

$$\bigcup_{b \in J_{4k+3},\ bR^2w} \mathrm{pr}(\pi^{-1}_{4k+3}(b)) \subseteq P \setminus \left(\bigcup_{b \in J_{4k+3},\ \neg(bR^3w)} \mathrm{pr}(\pi^{-1}_{4k+3}(b)) \right) \subseteq$$
$$\subseteq \mathrm{int}(\mathrm{cl}(U_3)) = \mathrm{int}(U_3) = U_3,$$

as U_3 is a regular open set, which finishes the proof of the lemma.

$\square_{\text{Claim5.10}}$

$\square_{\text{Lemma5.7}}$

The inductive steps (depending on the remainder of i modulo 4) will rely on Lemmas 5.11-5.14 below, with Lemma 5.11 handling the case of stepping from $4k$ to $4k+1$ (constructing J_{4k+1}, $\pi_{4k,4k+1}$, $h^\bullet_k$), and the last one handling the case of going from $4k+3$ to $4k+4$ (constructing J_{4k+4}, $\pi_{4k+3,4k+4}$, h_{k+1}). Before stating these lemmas we first prove some auxiliary lemmas about piecewise amalgamation with some values prescribed.

Lemma 5.11. *Suppose that $J \supseteq K$, $J' \supseteq K'$ are finite linear graphs, $\pi, h : J' \to J$ are epimorphisms, such that*

(1) $\pi \restriction K' = h \restriction K'$,

(2) $\pi[K'] = K$ $(= h[K'])$, π maps endpoints to endpoints,

(3) J' is the disjoint union of the distinct nonempty intervals C'_1, C'_2, K',

(4) $\pi[C'_1] = h[C'_1] = \pi[C'_2] = h[C'_2] = J$.

(5) for $c \in C'_1 \cup C'_2$, if vRK' (in particular, c is an endpoint of C'_j for $j = 1$ or $j = 2$), then

$$\mathrm{tp}^{c,C'_j}(h) = \mathrm{tp}^{c,C'_j}(\pi).$$

Then, there exist $J'' \supseteq K''$, and $\pi', h^\bullet \in \mathrm{Epi}(J'', J')$ that satisfy the following requirements:

$\boxplus_1$ *$h \circ h^\bullet = \pi \circ \pi'$,*

$\boxplus_2$ *J'' is the disjoint union of the nonempty intervals K'', C''_1, C''_2,*

$\boxplus_3$ *$h^\bullet \restriction K'' = \pi' \restriction K''$ with $\pi'[K''] = K'$, and these $(\pi' \restriction K'')$ map endpoints to endpoints,*

$\boxplus_4$ *$\pi'[C''_1] = h^\bullet[C''_1] = J'$, $\pi'[C''_2] = h^\bullet[C''_2] = J'$,*

$\boxplus_5$ *for $c \in C''_1 \cup C''_2$, if cRK'' (in particular, c is an endpoint of C_j for $j = 1$ or $j = 2$), then*

$$\mathrm{tp}^{c,C''_j}(h^\bullet) = \mathrm{tp}^{c,C''_j}(\pi').$$

$\boxplus_6$ *for each $d \in J'$, $(\pi')^{-1}(d)$ is the disjoint union of intervals, each of them is of length at least 3, in particular, aR^3b implies $\pi'(a)R\pi'(b)$.*

Proof. (Lemma 5.11) First we are going to amalgamate the pair h, π piecewise on C_1, on K and on C_2 to get J^+ which will be the disjoint union of K^+, C^+_1, C^+_2 (with $K^+RC^+_i$), the mappings $h^+, \pi^+ \in \mathrm{Epi}(J^+, J')$ with $\pi \circ \pi^+ = h \circ h^+$, such that $\pi^+[C^+_i] = h^+[C^+_i] = C'_i$, $h^+[K^+] = \pi^+[K^+] = K'$, and $h^+ \restriction K^+ = \pi^+ \restriction K^+$. This can be done by invoking the moreover part of Lemma B.1 two times, with the roles

- $M = C'_1 = M'$, $f = \pi \restriction C'_1$, $f' = h \restriction C'_1$, and setting $m_- = m'_-$ to be the node connected with K, after which we can let C^+_1 be the resulting O, $\pi^+ \restriction C^+_1$ be the resulting g, $h^+ \restriction C^+_1$ be the resulting g',
- $M = C'_2 = M'$, $f = \pi \restriction C'_2$, $f' = h \restriction C'_2$, and setting $m_- = m'_-$ to be the node connected with K, after which C^+_2 will be the resulting O.

We let K^+ be isomorphic to K and $\pi \restriction K = h^+ \restriction K$ be an isomorphism.

Now let J^+ be a linear graph which is the disjoint union of the intervals C^+_1, K^+, C^+_2 (where K^+ lies in the middle). Pick a finite linear graph J'' which is the disjoint union of the intervals C''_1, K'', C''_2 and $\pi^* \in \mathrm{Epi}(J'', J^+)$ such that $\pi^*[K''] = K^+$, $\pi^*[C''_i] = J^+$, moreover,

- if $C_1'' = \{a_0, a_1, \dots, a_{|C_1''|-1}\}$ where $K'' R a_0$, $a_i R a_{i+1}$, then the walk
$$\pi^*(a_0), \pi^*(a_1), \dots, \pi^*(a_{|C_1''|-1})$$
visits all nodes of C_1^+ before reaching K^+, i.e.
$$\mathrm{tp}^{a_0, C_1''}(\pi^*) \supseteq \{C_1^+, C_1^+ \cup K^+\},$$
- similarly, if $b_0 \in C_2''$, $b_0 R K''$, then
$$\mathrm{tp}^{b_0, C_2''}(\pi^*) \supseteq \{C_2^+, C_2^+ \cup K^+\},$$
- for every $d \in J^+$, each connected component of $(\pi^*)^{-1}(d)$ is an interval including at least 3 nodes.

One easily checks that $\pi' = \pi^* \circ \pi^+$, $h^\bullet = \pi^* \circ h^+$ will work.

$\square_{\text{Lemma5.11}}$

Lemma 5.12. *Suppose that $J \supseteq K$, $J' \supseteq K'$ are finite linear graphs, $\pi, h^\bullet : J' \to J$ are epimorphisms, such that*

(1) $\pi \restriction K' = h^\bullet \restriction K'$,
(2) $\pi[K'] = K$ $(= h^\bullet[K'])$, π maps endpoints to endpoints,
(3) J' is the disjoint union of the distinct nonempty intervals C_1', C_2', K',
(4) $\pi[C_1'] = h^\bullet[C_1'] = \pi[C_2'] = h^\bullet[C_2'] = J$.
(5) for $c \in C_1' \cup C_2'$, if cRK' (so then necessarily c is an endpoint of C_j' for $j = 1$ or 2), then
$$\mathrm{tp}^{c, C_j'}(h^\bullet) = \mathrm{tp}^{c, C_j'}(\pi).$$

Then, there exist $J'' \supseteq K'', L''$, and $\pi', h \in \mathrm{Epi}(J'', J')$ that satisfy the following requirements:

$\boxplus_1$ $h^\bullet \circ g = \pi \circ \pi'$,
$\boxplus_2$ J'' is the disjoint union of the nonempty intervals C_1'', L'', C_2'', K'', C_3'',
$\boxplus_3$ $g \restriction K'' = \pi' \restriction K''$ with $\pi'[K''] = K'$, and this epimorphism $(\pi' \restriction K'')$ maps endpoints to endpoints,
$\boxplus_4$ $g \restriction L'' = \pi' \restriction L''$ with $\pi'[L''] = K'$, and this epimorphism $(\pi' \restriction L'')$ maps endpoints to endpoints,
$\boxplus_5$ $\pi'[C_i''] = g[C_i''] = J'$ for $i = 1, 2, 3$,
$\boxplus_6$ for $c \in C_1'' \cup C_2'' \cup C_3''$, if $cR(K'' \cup L'')$ (so then necessarily c is an endpoint of C_j'' for $j \in \{1, 2, 3\}$), then
$$\mathrm{tp}^{c, C_j''}(g) = \mathrm{tp}^{c, C_j''}(\pi').$$
$\boxplus_7$ for each $d \in J'$, $(\pi')^{-1}(d)$ is the disjoint union of intervals, each of them is of length at least 3, in particular, aR^3b implies $\pi'(a) R \pi'(b)$.

Proof. Notice that the main difference compared to Lemma 5.11 is that we "double" K'' (and we have 3 complementer intervals instead of 2). Let J^*, K^*, C_1^*, C_2^*, π^*, g^* be J'', K'', C_1'', C_2'', π'' and $h^\bullet$ given by Lemma 5.11. Our J'' will be an appropriate extension of J^+.

So let J'' be a finite linear graph with an epimorphism $\pi^{**} : J'' \to J^*$ that satisfies

- J'' is the disjoint union of the connected subgraphs C_1'', L'', C_2'', K'', C_3'', where consecutive ones are connected,

- $\pi^{**}[C_1''] = \pi^{**}[C_3''] = C_1^*$,
- $\pi^{**}[C_2''] = C_2^*$,
- $\pi^{**}[K''] = \pi^{**}[L''] = K^*$,
- for every $d \in J^*$, each connected component of $(\pi^{**})^{-1}(d)$ is an interval including at least 3 nodes.

It is straightforward to check that $\pi' = \pi^* \circ \pi^{**}$, $g = g^* \circ \pi^{**}$ work. $\Box_{\text{Lemma5.12}}$

Lemma 5.13. *Suppose that $J \supseteq K, L$, $J' \supseteq K', L'$ are finite linear graphs, $\pi, g : J' \to J$ are epimorphisms, such that*

(1) $\pi \restriction K' = g \restriction K'$,
(2) $\pi[K'] = K$ $(= g[K'])$, π maps endpoints to endpoints,
(3) $\pi \restriction L' = g \restriction L'$,
(4) $\pi[L'] = K$ $(= g[K'])$, π maps endpoints to endpoints,
(5) J' is the disjoint union of the distinct nonempty intervals C_1', L', C_2', K', C_3' (with consecutive ones being connected to each other),
(6) $\pi[C_i'] = g[C_i'] = C_1$ for $i = 1, 2, 3$,
(7) for $x \in C_i'$, if xRK', or xRL' (so then necessarily x is an endpoint of C_i'), then

$$\mathrm{tp}^{x, C_i'}(g) = \mathrm{tp}^{x, C_i'}(\pi).$$

Then, there exist $J'' \supseteq K'', L''$, and $\pi', f, f' \in \mathrm{Epi}(J'', J)$ that satisfy the following requirements:

$\boxplus_1$ *$g \circ \pi' = \pi \circ f'$,*
$\boxplus_2$ *J'' is the disjoint union of the distinct nonempty intervals C_1'', L'', C_2'', K'', C_3'', with the consecutive ones being connected,*
$\boxplus_3$ *$f' \restriction K'' = \pi' \restriction K''$ with $\pi'[K''] = K'$,*
$\boxplus_4$ *$f' \restriction L'' = \pi' \restriction L''$ with $\pi'[L''] = L'$,*
$\boxplus_5$ *$\pi'[C_i''] = C_i' = f'[C_i'']$ for $i = 1, 2, 3$,*
$\boxplus_6$ *for each $d \in J'$, $(\pi')^{-1}(d)$ is the disjoint union of intervals, each of them is of length at least 3, in particular, aR^3b implies $\pi'(a)R\pi'(b)$.*
$\boxplus_7$ *for $a \in J'' \setminus L''$ we have $f(a) = f'(a)$,*
$\boxplus_8$ *for each $a \in L''$ we have $|f(a) - f'(a)| \leq 2$, and if $a \in L''$ is such that $f'(a) = \pi'(a)$ is neither an endpoint of L' nor is related to one then $|f(a) - f'(a)| = 2$,*

Proof. We are going to construct

- K'', $\pi' \restriction K''$, $f' \restriction K'''$,
- L'', $\pi' \restriction L''$, $f' \restriction L''$,
- C_i'', $\pi' \restriction C_i''$, $f' \restriction C_i''$ for $i = 1, 2, 3$

separately, in such a way that they map endpoints to endpoints. If these restrictions satisfy $\boxplus_1$, then we will have $\boxplus_1$-$\boxplus_5$.

We can apply the moreover part of Lemma B.1 separately to the pairs $g \restriction C_i'$ and $\pi \restriction C_i'$ (so $M = M' = C_i'$, $f = g \restriction C_i'$, $f' = \pi \restriction C_i'$) where $i = 1, 3$, to obtain C_i'', $f' \restriction C_i''$, $\pi' \restriction C_i''$.

Next we apply the main part of Lemma B.1 to C_2. Finally, since $\pi \restriction K' = g \restriction K'$, and $\pi \restriction L' = g \restriction L'$ we can take $K'' = K'$, $L'' = L'$ (and $f' \restriction K'' = \pi' \restriction K'' = \mathrm{id}_{K''}$).

Now $J'' = C_1'' \cup L'' \cup C_2'' \cup K'' \cup C_3''$ (connecting the pieces in this order) will satisfy $\boxplus_1$-$\boxplus_5$. If $\pi^* : J^* \to J''$ is an epimorphism where each point's preimage is exactly an interval containing 3 nodes, then replacing f', π' with $f' \circ \pi^*$, $\pi' \circ \pi^*$ and replacing J'' with J^* will ensure $\boxplus_6$, too.

Finally, to get f we first note that $(f')^{-1}(L') = L''$ implies that the endpoints of L'' are sent to those of L', i.e. if $l''_-, l''_+ \in L''$ are such that $l''_- R C_1''$, $l''_+ R C_2''$, and the endpoints $l'_-, l'_+ \in L'$ are such that $l'_- R C_1'$, $l'_+ R C_2'$, then $f'(l''_-) = l'_-$, $f'(l''_+) = l'_+$ (for this we used also $f'[C_i''] = C_i'$). We also note that by $\boxplus_6$ there is a connected subgraph of L'' containing l''_-, consisting of at least 3 elements all of which are mapped to l'_- by f'.

Identifying L' with $\{0, 1, 2, \ldots, |L'| - 1\}$ (where $l'_- = 0$, $l'_+ = |L'| - 1$, we can define $f \restriction L''$ as follows.

$$f \restriction L''(a) = \left\{ \begin{array}{ll} f'(a) \ (= 0) & \text{if } a = l''_- \\ f'(a) + 1 \ (= 1) & \text{if } a \neq l''_-, \ a R l''_- \\ \max(f'(a) + 2, |L'| - 1), & \text{otherwise.} \end{array} \right\}$$

Finally, letting

$$f \restriction J'' \setminus L'' = f' \restriction J'' \setminus L'',$$

it is straightforward to check that $f \in \mathrm{Epi}(J'', J')$ with the desired properties.

$\square_{\text{Lemma5.13}}$

Lemma 5.14. *Suppose that $J \supseteq K, L$, $J' \supseteq K', L'$, $\pi, f : J' \to J$ are epimorphisms, such that*

(1) $\pi \restriction K' = f \restriction K'$,
(2) $\pi[K'] = K$ *($= f[K']$), π maps endpoints to endpoints,*
(3) $\pi \restriction L' = f \restriction L'$,
(4) $\pi[L'] = K$ *($= f[K']$), π maps endpoints to endpoints,*
(5) *J' is the disjoint union of the distinct nonempty intervals C_1', L', C_2', K', C_3' (with consecutive ones being connected to each other), similarly, J is the disjoint union of C_1, L, C_2, K and C_3,*
(6) $\pi[C_i'] = f[C_i'] = C_i$ *for* $i = 1, 2, 3$,

Assume moreover, that

(7) *M is a finite linear graph, $\phi \in \mathrm{Epi}(M, J')$.*

Then, there exist $J'' \supseteq K''$, and $\pi', h \in \mathrm{Epi}(J'', J')$ that satisfy the following requirements:

$\boxplus_1$ $f \circ \pi' = \pi \circ h$,
$\boxplus_2$ *there exists $\phi^* \in \mathrm{Epi}(J'', M)$ with $\phi \circ \phi^* = \pi'$,*
$\boxplus_3$ *J'' is the disjoint union of C_1'', K'', C_2'' (consecutive ones are connected),*
$\boxplus_4$ $\pi' \restriction K'' = h \restriction K''$,
$\boxplus_5$ $\pi'[K''] = K' (= h[K''])$,
$\boxplus_6$ *for* $i = 1, 2$

$$\pi'[C_i''] = h[C_i''] = J',$$

and if $c \in C_i''$ satisfies cRK'', then

$$\mathrm{tp}^{c, C_i''}(h) = \mathrm{tp}^{c, C_i''}(\pi').$$

$\boxplus_7$ *for each $d \in J'$, $(\pi')^{-1}(d)$ is the disjoint union of intervals, each of them is of length at least 3, in particular, aR^3b implies $\pi'(a)R\pi'(b)$.*

Proof. The finite linear graph J'' will be the result of three successive extension of J'. We are going to define

- J^* and $\pi^*, h^* \in \mathrm{Epi}(J^*, J')$ with $f \circ \pi' = \pi \circ h^*$,
- J^{**} and $\pi^{**} \in \mathrm{Epi}(J^{**}, J^*)$,
- and finally J'', $\pi'' \in \mathrm{Epi}(J'', J^*)$,

and we will let $\pi' = \pi^* \circ \pi^{**} \circ \pi''$, $h = h^* \circ \pi^{**} \circ \pi''$.

The extension J^* will be the disjoint union of the finite linear graphs C_1^*, L^*, C_2^*, K^*, C_3^*, which are connected to each other in this order. C_1^*, $\pi^* \restriction C_1^*$, $h^* \restriction C_1^*$ are gotten by applying Lemma B.1 to $g \restriction C_1$ and $\pi \restriction C_1$, and similarly, we obtain $D \in \{L^*, C_2^*, C_3^*\}$, $\pi^* \restriction D$, $h^* \restriction D$ by the same way, while we can let K^* be isomorphic to K' (since $f \restriction K' = \pi \restriction K'$), so

$\blacksquare_1$ for $D \in \{C_1^*, L^*, C_2^*, K^*, C_3^*\}$) $\pi^*[D] = h^*[D]$, and π^*, h^* map endpoints of D to endpoints of $\pi^*[D]$ (which is C_1^* if $D = C_1^*$, etc.),

$\blacksquare_2$ $\pi^* \restriction K^* = h^* \restriction K^*$,

$\blacksquare_3$ in particular, if $D \neq E \in \{C_1^*, L^*, C_2^*, K^*, C_3^*\}$, then $\pi^*[D] \cap \pi^*[E] = \emptyset$,

Next, we define an extension J^{**} of J^* as follows. The pair J^{**} and $\pi^{**} \in \mathrm{Epi}(J^{**}, J^*)$ is in fact uniquely determined (up to isomorphism) by the requirements that

- J^{**} is the disjoint union of D_i, $i = 1, 2, \ldots, 8$ with $D_i R D_{i+1}$,
- for each i the map $\pi^{**} \restriction D_i$ is a bijection,
- $\pi^{**}[D_1] = \pi^{**}[D_8] = J^*$,
- $\pi^{**}[D_2] = \pi^{**}[D_7] = C_3^*$,
- $\pi^{**}[D_i] = K^*$ for $i = 3, 4, 5, 6$.

For future reference we remark that

$$(\pi^{**} \restriction D_8 \text{ is an isomorphism with } J^*) \ \& \ (d \in D_8 \wedge dRD_7 \ \to \pi^{**}(d) \in C_3^*), \tag{5.23}$$

$$(\pi^{**} \restriction D_1 \text{ is an isomorphism with } J^*) \ \& \ (d \in D_1 \wedge dRD_2 \ \to \pi^{**}(d) \in C_3^*), \tag{5.24}$$

Next, we are going to use M and f from (7) to construct J''. W.l.o.g. we can assume that f factors through $\pi^* \circ \pi^{**}$, or simply f maps onto J^{**}, so we can pick J'', $f^* \in \mathrm{Epi}(J'', M)$, $\pi'' \in \mathrm{Epi}(J'', J^{**})$ with $\pi'' = f \circ f^*$. We can assume that $(\pi'')^{-1}(a)$ consists of intervals each of length at least 3 whenever $a \in J^{**}$.

Finally, we claim that letting $\pi' = \pi^* \circ \pi^{**} \circ \pi''$, $h = h^* \circ \pi^{**} \circ \pi''$, and K'' be any interval in J'' with

- $\pi''[K''] = D_4 \cup D_5$, and K'' is maximal such interval (i.e. $a \notin K''$, aRK'' implies $\pi''(a) \in D_3 \cup D_6$),
- if $C \subseteq J'' \setminus K''$ is maximal connected, then $\pi'' \restriction C$ is surjective

will work (i.e. $\boxplus_6$ is satisfied). We also remark that passing to a further extension of J'', if necessary and replacing J'', π'', without loss of generality such K'' exists.

So let $C \subseteq J'' \setminus K''$ be a maximal connected subgraph, $c \in C$ be such that cRK'', we need to check that

$$\mathrm{tp}^{c,C}(h) = \mathrm{tp}^{c,C}(\pi').$$

It is easy to see that $\pi''(c) \in D_3 \cup D_6$. We let k^*_- denote the unique node in K^* with $k^*_- R C^*_2$. By the way $\pi^{**} : J^{**} \to J^*$ is defined one can check that

$$\pi^{**} \circ \pi''(c) = k^*_-.$$

We claim that

$$\text{(5.25)}\quad \mathrm{tp}^{c,C}(\pi^{**} \circ \pi'') \supseteq \{\{k^*_-\}, K^*, K^* \cup C^*_3, K^* \cup C^*_3 \cup C^*_2, K^* \cup C^*_3 \cup C^*_2 \cup L^*\}.$$

Clearly $\mathrm{tp}^{c,C}(\pi'')$ contains a set H, such that H is either the entire D_3 and a subset of $D_4 \cup D_5 \cup D_6$, or H contains the entire D_6, and a subset of $D_3 \cup D_4 \cup D_5$. Either case $\pi^{**}[H] = K^*$. Similarly, $\mathrm{tp}^{c,C}(\pi'')$ contains a set H' that is either the union of $D_2 \cup D_3$ and a subset of $D_4 \cup D_5 \cup D_6 \cup D_7$, or H is the union of the full $D_6 \cup D_7$ and a subset $D_2 \cup D_3 \cup D_4 \cup D_5$. In either case, $\pi^{**}[H] = K^* \cup C^*_3$. This implies that

$$\mathrm{tp}^{c,C}(\pi^{**} \circ \pi'') \supseteq \{\{k^*_-\}, K^*, K^* \cup C^*_3\}.$$

Now recalling (5.23) and (5.24) it is easy to check that (5.25) holds, indeed. Finally, k^*_- is an endpoint of K^*, so by $\blacksquare_1$ and $\blacksquare_3$ the node $\pi^*(k^*_-) = \pi'(c) = h^*(k^*_-) = h'(c)$ is an endpoint of K', so applying π^* and h^* to (5.25),

$$\mathrm{tp}^{c,C}(\pi^* \circ \pi^{**} \circ \pi'') \supseteq \{\{\pi'(k''_-)\}, K', K' \cup C'_3, K' \cup C'_3 \cup C'_2, K' \cup C'_3 \cup C'_2 \cup L'\},$$

and

$$\mathrm{tp}^{c,C}(h^* \circ \pi^{**} \circ \pi'') \supseteq \{\{\pi'(k''_-) = h^*(k^*_-)\}, K', K' \cup C'_3, K' \cup C'_3 \cup C'_2, K' \cup C'_3 \cup C'_2 \cup L'\},$$

which by our assumption (5) on the structure of J' uniquely determine $\mathrm{tp}^{c,C}(\pi^* \circ \pi^{**} \circ \pi'')$ and $\mathrm{tp}^{c,C}(h^* \circ \pi^{**} \circ \pi'')$, in particular these two coincide. $\square_{\text{Lemma5.14}}$

$\square_{\text{Proposition5.4}}$

Appendix A. Projective Fraïssé limits

We recall here the framework of projective Fraïssé theory. The theory was introduced in [7]. The description below is a generalized version of [7].

Category $\mathcal{K}$. We fix a symbol R. By an **interpretation of** R on a set X we understand a binary relation R^X on X, that is, $R^X \subseteq X \times X$. We say that R^X is a **reflexive graph** if it is reflexive and symmetric as a binary relation. Assume X and Y are equipped with interpretations R^X and R^Y of R. Then a function $f\colon X \to Y$ is called a **strong homomorphism** if

— for all $x_1, x_2 \in X$, $x_1 R^X x_2$ implies $f(x_1) R^Y f(x_2)$;
— for all $y_1, y_2 \in Y$, $y_1 R^Y y_2$ implies that there exist $x_1, x_2 \in X$ with $y_1 = f(x_1)$, $y_2 = f(x_2)$, and $x_1 R^X x_2$.

By reflexivity of R^Y, a strong homomorphism is a surjective function from X to Y. We will often skip the superscript X in R^X trusting that the context determines which interpretation of R we have in mind.

We have a category $\mathcal{K}$ each of whose objects is a finite set equipped with an interpretation of R as a reflexive graph, all of whose morphisms are strong homomorphisms, and the following condition holds

(o) if A, B, C are objects in $\mathcal{K}$, $f\colon A \to B$ and $h\colon A \to C$ are morphisms in $\mathcal{K}$, and $g\colon B \to C$ is a function such that $h = g \circ f$, then g is a morphism in $\mathcal{K}$.

We say that $\mathcal{K}$ is a **projective Fraïssé class** if it fulfills the following two conditions:

(F1) for any two objects A, B in $\mathcal{K}$, there exist an object C in $\mathcal{K}$ and morphisms $f\colon C \to A$ and $g\colon C \to B$ in $\mathcal{K}$;

(F2) for any two morphisms f, g in $\mathcal{K}$ with the same codomain, there exist morphisms f', g' in $\mathcal{K}$ such that $f \circ f' = g \circ g'$.

Condition (i) is called the **joint projection property** and condition (ii) is called the **projective amalgamation property**.

Category $\mathcal{K}^*$ and projective Fraïssé limit. We consider a category $\mathcal{K}^*$ whose objects are totally disconnected compact metric spaces $\mathbb{K}$ taken together with sets $\mathrm{Epi}(\mathbb{K}, A)$ of continuous surjective functions from $\mathbb{K}$ to A, for $A \in \mathcal{K}$, with the following properties:

(A1) for each continuous function $\psi\colon \mathbb{K} \to X$, where X is a finite topological space, there exist $\varphi \in \mathrm{Epi}(\mathbb{K}, A)$ and a function $f\colon A \to X$, for some $A \in \mathcal{K}$, such that
$$\psi = f \circ \varphi;$$

(A2) for all $\phi \in \mathrm{Epi}(\mathbb{K}, A)$ and $\psi \in \mathrm{Epi}(\mathbb{K}, B)$, for some $A, B \in \mathcal{K}$, there exists $\chi \in \mathrm{Epi}(\mathbb{K}, C)$, $f \in \mathrm{Epi}(C, A)$, and $g \in \mathrm{Epi}(C, B)$, for some $C \in \mathcal{K}$, such that
$$\phi = f \circ \chi \text{ and } \psi = g \circ \chi;$$

(A3) for a strong homomorphism $f\colon A \to B$, for some $A, B \in \mathcal{K}$, we have
$$\begin{aligned}\big(f \circ \varphi \in \mathrm{Epi}(\mathbb{K}, B),\ \text{for some } \varphi \in \mathrm{Epi}(\mathbb{K}, A)\big) &\Rightarrow f \in \mathrm{Epi}(A, B)\\ &\Rightarrow \big(f \circ \varphi \in \mathrm{Epi}(\mathbb{K}, B),\ \text{for all } \varphi \in \mathrm{Epi}(\mathbb{K}, A)\big).\end{aligned}$$

Conditions (A1), (A2), (A3) assert that elements of $\mathcal{K}^*$, that is, $\mathbb{K}$ together with $\mathrm{Epi}(\mathbb{K}, A)$, for $A \in \mathcal{K}$, can be viewed as inverse limits of sequences consisting of structures in $\mathcal{K}$ with bonding maps being morphisms in $\mathcal{K}$.

A morphism in $\mathcal{K}^*$ is a continuous surjection $\sigma\colon \mathbb{K} \to \mathbb{K}'$ such that $\varphi \circ \sigma \in \mathrm{Epi}(\mathbb{K}, A)$, for each $\varphi \in \mathrm{Epi}(\mathbb{K}', A)$ with $A \in \mathcal{K}$. By
$$\mathrm{Aut}(\mathbb{K})$$
we denote the group of all invertible morphisms $\mathbb{K} \to \mathbb{K}$, that is, all homeomorphisms $\sigma\colon \mathbb{K} \to \mathbb{K}$ such that both it and σ^{-1} are morphisms in $\mathcal{K}^*$.

Proposition A.1. *Let $\mathcal{K}$ be a countable projective Fraïssé class. There exists an object $\mathbb{K}_\infty$ in $\mathcal{K}^*$ such that*

(P1) *(projective universality)* $\mathrm{Epi}(\mathbb{K}_\infty, A) \neq \emptyset$, *for each $A \in \mathcal{K}$,*

(P2) *(projective ultrahomogeneity) for each $A \in \mathcal{K}_\infty$, $\varphi, \psi \in \mathrm{Epi}(\mathbb{K}_\infty, A)$ there exists $\sigma \in \mathrm{Aut}(\mathbb{K}_\infty)$ such that $\varphi = \psi \circ \sigma$.*

The object in $\mathcal{K}^*$ with properties (P1) *and* (P2) *is unique up to isomorphism.*

The unique object $\mathbb{K}_\infty$ in Proposition A.1 above is called the **projective Fraíssé limit** of $\mathcal{K}$.

The proposition is proved by constructing structures A_n in $\mathcal{K}$ and $\pi_n\colon A_{n+1} \to A_n$ in $\mathrm{Epi}(A_{n+1}, A_n)$, $n \in \mathbb{N}$, such that

(G1) for each $A \in \mathcal{K}$, there exists $m \in \mathbb{N}$ with $\mathrm{Epi}(A_m, A) \neq \emptyset$;

(G2) for each $A \in \mathcal{K}$ and $f \in \mathrm{Epi}(A, A_m)$, for some $m \in \mathbb{N}$, there exists $n > m$ and $g \in \mathrm{Epi}(A_n, A)$ such that

$$f \circ g = \pi_m \circ \cdots \circ \pi_{n-1}.$$

The construction is done by induction using countability of $\mathcal{K}$ and properties (F1) and (F2). One lets $\mathbb{K}_\infty = \operatorname{proj\,lim}_n(A_n, \pi_n)$. One then has the canonical projection maps $\pi_n^\infty\colon \mathbb{K}_\infty \to A_n$, and defines, for $A \in \mathcal{K}$,

$$\mathrm{Epi}(\mathbb{K}_\infty, A) = \{f \circ \pi_n^\infty \mid f \in \mathrm{Epi}(A_n, A) \text{ for some } n \in \mathbb{N}\}.$$

A sequence with properties (G1) and (G2) is called a **generic sequence** for $\mathcal{K}$. One checks properties (A1)–(A3) and (P1), (P2). Property (o) of the Fraïssé class $\mathcal{K}$ is not used in the construction; it is used to check the first implication in (A3).

Define a binary relation $R^{\mathbb{K}}$ on any $\mathbb{K} \in \mathcal{K}^*$ by letting

$$xR^{\mathbb{K}}y \text{ iff } \big(\varphi(x)R^A\varphi(y), \text{ for all } A \in \mathcal{K} \text{ and all } \varphi \in \mathrm{Epi}(\mathbb{K}, A)\big).$$

We note that $R^{\mathbb{K}}$ is a compact symmetric and reflexive binary relation on $\mathbb{K}$. We also note that all elements of $\mathrm{Aut}(\mathbb{K})$ are isomorphisms of the structure $(\mathbb{K}, R^{\mathbb{K}})$.

The canonical quotient space of a transitive class $\mathcal{K}$. We will abandon the subscript in the notation $\mathbb{K}_\infty$ for the projective Fraïssé limit of $\mathcal{K}$.

We say that the Fraïssé class $\mathcal{K}$ is **transitive** if $R^{\mathbb{K}}$ is a transitive relation on the projective Fraïssé limit $\mathbb{K}$ of $\mathcal{K}$. Transitivity of $R^{\mathbb{K}}$ implies that it is a compact equivalence relation on $\mathbb{K}$ since $R^{\mathbb{K}}$ is compact, symmetric, and reflexive by its very definition.

Assume $\mathcal{K}$ is a transitive projective Fraïssé class. Then

$$K = \mathbb{K}/R^{\mathbb{K}}$$

with the quotient topology is a compact metric space. We call it the **canonical quotient space** of $\mathcal{K}$. Let

$$\mathrm{pr}\colon \mathbb{K} \to K$$

be the quotient map, which we call the projection.

Fact A.2. *If $x \in \mathbb{K}$, then sets of the form*

$$\{\mathrm{pr}(y) : \ y \in \mathbb{K}, \ \varphi(x)R\varphi(y)\}, \ \varphi \in \mathrm{Epi}(\mathbb{K}, A), \ A \in \mathcal{K}$$

form a neighborhood basis of $\mathrm{pr}(x) \in K$.

Proof. For a fixed $\varphi \in \mathrm{Epi}(\mathbb{K}, A)$ let $A_0 \subseteq A$ be $A_0 = \{a \in A : \ \varphi(x)Ra\}$. Clearly

$$\{\mathrm{pr}(y) : \ y \in \mathbb{K}, \ \varphi(y) \in A_0\} \supseteq K \setminus \{\mathrm{pr}(y) : \ y \in \mathbb{K}, \ \varphi(y) \notin A_0\} \not\ni \mathrm{pr}(x)$$

since for any $y \in \mathbb{K}$ $\neg(\varphi(y)R\varphi(x))$ implies $\neg xRy$, so $\mathrm{pr}(x) \neq \mathrm{pr}(y)$. As $\{\mathrm{pr}(y) : \ y \in \mathbb{K}, \ \varphi(y) \notin A_0\}$ is closed, $\{\mathrm{pr}(y) : \ y \in \mathbb{K}, \ \varphi(y) \in A_0\}$ is a neighborhood of $\mathrm{pr}(x)$, indeed.

Suppose that $U \subseteq K$ is open and $\mathrm{pr}(x) \in U$. Since there are only countably many epimorphisms in $\mathcal{K}$, by the amalgamation property of $\mathcal{K}$ (and by (P1), (P2)), we can enumerate a cofinal system $(\varphi_i)_i$ of epimorphisms, in the sense that each epimorphism factors through all, but finitely many φ_i. For each φ_i pick $y_i \in \mathbb{K}$ with $\mathrm{pr}(y_i) \notin U$, $\varphi(y_i) R \varphi(x)$. By the previous remark about $(\varphi_i)_i$ (i.e. for each $i \leq j$ there exists π such that $\varphi_i = \pi \circ \varphi_j$) and by a standard compactness argument we can assume $\varphi_i(y_j) = \varphi_i(y_i))$ for $i \leq j$. By compactness of $\mathbb{K}$, we can assume that $(y_i)_i$ converges to some $y \in \mathbb{K}$ (in fact the convergence of $(\varphi_i(y_j))_j$ already implies this), so by continuity,

$$\text{for each } i,\ \varphi_i(y) = \varphi_i(\lim_j(y_j)) = \lim_j(\varphi_i(y_j)) = \varphi_i(y_i),$$

in particular, $\varphi_i(y) R \varphi_i(x)$. Since every $\varphi \in \mathrm{Epi}(\mathbb{K}, A)$ factors through some φ_i via an epimorphism we obtain that $\varphi(x) R \varphi(y)$ holds (for every φ). So $x R^{\mathbb{K}} y$, and $\mathrm{pr}(x) = \mathrm{pr}(y) = \lim_i \mathrm{pr}(y_i) \in \mathrm{cl}(K \setminus U) = K \setminus U$, contradicting that $\mathrm{pr}(x) \in U$ is open. $\square_{\text{FactA.2}}$

There is a natural continuous homomorphism

$$\mathrm{pr}\colon \mathrm{Aut}(\mathbb{K}) \to \mathrm{Homeo}(K)$$

induced by the projection $\mathrm{pr}\colon \mathbb{K} \to K$. Namely, given $f \in \mathrm{Aut}(\mathbb{K})$ and $x \in K$, we fix $p \in \mathbb{K}$ with $x = \mathrm{pr}(p)$ and let

$$\mathrm{pr}(f)(x) = \mathrm{pr}\big(f(p)\big).$$

It is now easy to check that, since f is an automorphism of $\mathbb{K}$, the value of $\mathrm{pr}(f)(x)$ does not depend on the choice of p. It is also easy to see that $\mathrm{pr}(f)$ is continuous and bijective, so it is a homeomorphism of K. Continuity of $\mathrm{pr}\colon \mathrm{Aut}(\mathbb{K}) \to \mathrm{Homeo}(K)$ is then easy to check. It is clear that, for $f, g \in \mathrm{Aut}(\mathbb{K})$, we have

$$\mathrm{pr}(f) = \mathrm{pr}(g) \ \text{ iff } \ \forall x, y \in \mathbb{K}\ \big(x R^{\mathbb{K}} y \Rightarrow f(x) R^{\mathbb{K}} g(y)\big). \tag{A.1}$$

Appendix B. An amalgamation lemma

In this section, we state and prove an amalgamation lemma, which is a special case of a result from an unpublished work by Solecki and Tsankov. We will need this lemma for the construction in Section 5. For the sake of completeness, we provide a direct proof of this particular case here.

Lemma B.1 (Solecki–Tsankov)**.** *Suppose that*

(1) *L is a finite linear graph, M, M' denote finite linear graphs,*
(2) *$f : M \to L$, $f' : M' \to L$ are epimorphisms,*
(3) *m_-, m_+ (m'_-, m'_+, resp.) are endpoints of M (M', resp.) which satisfy*

$$\mathrm{tp}^{m_-,M}(f) = \mathrm{tp}^{m'_-,M'}(f').$$

$$\mathrm{tp}^{m_+,M}(f) = \mathrm{tp}^{m'_+,M'}(f').$$

Then there exist

- *a finite linear graph O with endpoints o_-, o_+,*

- *and epimorphisms* $g : O \to M$, $g' : O \to M'$ *with*

$$f \circ g = f' \circ g',$$

  and
- $g(o_-) = m_-$,
- $g'(o_-) = m'_-$,
- $g(o_+) = m_+$,
- $g'(o_-) = m'_+$.

Moreover, if we only assume $\mathrm{tp}^{m_-,M}(f) = \mathrm{tp}^{m'_-,M'}(f')$, *then there exists* $g : O \to M$, $g' : O \to M'$ *with*

$$f \circ g = f' \circ g',$$

and an endpoint o_- *such that* $g(o_-) = m_-$, $g'(o_-) = m'_-$.

First, we deal with the following particular case of Lemma B.1.

Lemma B.2. *Suppose that* L, M, M' *are finite linear graphs,* $f \in \mathrm{Epi}(M, L)$, $f' \in \mathrm{Epi}(M', L)$, *and the endpoints* l_+, l_- *of* L, m_+, m_- *of* M, m'_+, m'_- *of* M' *satisfy* $f(m_-) = l_-$, $f(m_+) = l_+$, $f'(m'_-) = l_-$, $f'(m'_+) = l_+$.

Then, for some finite linear graph O *and epimorphisms* $g : O \to M$, $g' : O \to M'$ *we have* $f \circ g = f' \circ g'$, *and one endpoint of* O *is mapped to* m_- *by* g *(*m'_- *by* g'*, resp.), while the other endpoint is mapped to* m_+ *and* m'_+.

Proof. Extend the linear graphs M, M', L by two points as follows. We let

$$M_{ext} = M \cup \{d_-, d_+\},$$

such that d_- is connected with m_-, and d_+ is connected with $cm+$. Similarly,

$$M'_{\mathrm{ext}} = M' \cup \{d'-, d'_+\},$$

and

$$L_{\mathrm{ext}} = L \cup \{l^*_-, l^*_+\}.$$

It is easy to see that $\tilde{f} : M_{\mathrm{ext}} \to L_{\mathrm{ext}}$, defined by the identities $\tilde{f}' \restriction M = f$,

$$\begin{aligned} \tilde{f}(d_-) &= l^*_- \\ \tilde{f}(d_+) &= l^*_+ \end{aligned} \tag{B.1}$$

is an epimorphism. Similarly, letting $\tilde{f} : M \to L_{\mathrm{ext}}$ denote the mapping that satisfies $\tilde{f}' \restriction M' = f'$,

$$\begin{aligned} \tilde{f}'(d'_-) &= l^*_-, \\ \tilde{f}'(d'_+) &= l^*_+ \end{aligned} \tag{B.2}$$

is an epimorphism extending f'. Now let O be a linear graph for which for some epimorphisms $\tilde{g} : O \to M_{\mathrm{ext}}$, and $\tilde{g}' : O \to M'_{\mathrm{ext}}$ we have $\tilde{f} \circ \tilde{g} = \tilde{f}' \circ \tilde{g}'$. By passing to a minimal subinterval of O with $\tilde{f} \circ \tilde{g} (= \tilde{f}' \circ \tilde{g}')$ is surjective we can assume that endpoints of O are mapped to endpoints of L_{ext} (i.e. to l^*_-, l^*_+) by $\tilde{f} \circ \tilde{g}$, and only those are mapped to the endpoints of L_{ext}.

Now it is easily checked that recalling (B.1), (B.2) and $l^*_-, l^*_+ \in L_{\mathrm{ext}} \setminus L$, for every $a \in O$

$$\tilde{f} \circ \tilde{g}(a) = l^*_- \iff \tilde{g}(a) = d_-,$$

and

$$\tilde{f}' \circ \tilde{g}'(a) = l^*_- \iff \tilde{g}'(a) = d'_-.$$

Again by minimality one easily checks that if $a \in O$ is adjacent to an endpoint of O, then

- either $\tilde{f} \circ \tilde{g}(a) = \tilde{f}' \circ \tilde{g}'(a) = l_-$, and $\tilde{f}(a) = m_-$, $\tilde{f}'(a) = m'_-$,
- or $\tilde{f} \circ \tilde{g}(a) = \tilde{f}' \circ \tilde{g}'(a) = l_+$, and $\tilde{f}(a) = m_+$, $\tilde{f}'(a) = m'_+$

(and similarly with l^*_+, m_+, m'_+). So replacing O with the subgraph that remains upon removal of the two endpoints works. $\square_{\text{LemmaB.2}}$

The general case of Lemma B.1 will be reduced to the lemma above through the following lifting argument.

Lemma B.3. *Let L be a finite linear graph, l^+, l^- be two nodes and t^+, t^- be two L-types in the sense of Definition 5.5 starting with l^+, l^-, respectively, that is, t^- is of the form $\{\{l^-_i : i < k\} : k = 1, 2, \dots, |L|\}$, an increasing chain of connected subgraphs of L with $l_0 = l^-$ and $\{l^-_i : i < |L|\} = L$.*

Then there exists a finite linear graph L^*, and epimorphism $h \in \mathrm{Epi}(L^*, L)$, with endpoints l^*_-, l^*_+ of L^* satisfying $\mathrm{tp}^{l^*_-, L^*}(h) = t^-$, $\mathrm{tp}^{l^*_+, L^*}(h) = t^+$, such that whenever M is a finite linear graph with endpoints $m-, m_+ \in M$ and $f \in \mathrm{Epi}(M, L)$ that satisfies $\mathrm{tp}^{m_+, M}(f) = t^+$, $\mathrm{tp}^{m_-, M}(f) = t^-$, then there exists a finite linear graph M^*, endpoints $m^*_+, m^*_- \in M^*$, epimorphisms $g \in \mathrm{Epi(M^*, M)}$ $f^* \in \mathrm{Epi}(M^*, L^*)$ with

- *$f \circ g = h \circ f^*$,*
- *$g(m^*_+) = m_+$,*
- *$g(m^*) = m_-$,*
- *$f^*(m^*_+) = l^*_+$,*
- *$f^*(m^*_-) = l^*_-$,*

(in particular both f^* and g map endpoints to endpoints).

Proof. The lifting will be done in two steps, handling only one endpoint at a time.

Claim B.4. *Assume that L, t^+, t^- are as in the lemma. Then there exists a finite linear graph L^*, an epimorphism $h \in \mathrm{Epi}(L^*, L)$, an endpoint l^*_-, and an L^*-type*

$$(t^*)^+ = \{\{(l^*)^+_j : j < k\} : k \leq |L^*|\},$$

such that $h((t^*)^+) = t^+$, moreover, if l^+_0 is an endpoint of L, then so is $(l^*)^+_0$, and $(l^*)^+_0 \neq l^*_-$.

Moreover, whenever M is a finite linear graph with endpoints $m-, m_+ \in M$ and $f \in \mathrm{Epi}(M, L)$ that satisfies $\mathrm{tp}^{m_+, M}(f) = t^+$, $\mathrm{tp}^{m_-, M}(f) = t^-$, then there exists a finite linear graph M^*, endpoints $m^*_+, m^*_- \in M^*$, epimorphisms $g \in \mathrm{Epi(M^*, M)}$ $f^* \in \mathrm{Epi}(M^*, L^*)$ with

- *$f \circ g = h \circ f^*$,*
- *$g(m^*_+) = m_+$,*
- *$g(m^*) = m_-$,*
- *$\mathrm{tp}^{m^*_+, M^*}(f^*) = (t^*)^+$,*
- *$f^*(m^*_-) = l^*_-$ (in particular, $\mathrm{tp}^{m^*_-}(f^*)$ is the unique L^*-type starting with l^*_-).*

First we construct L^* and $h : L^* \to L$. The linear graph L^* will be the disjoint union of the linear graphs P, Q and S with

(∇_P) $P = \{p_0, p_1, \ldots, p_{|P|-1}\}$,
- $(\nabla_P)_1$ p_i is connected with p_j iff $|i-j| \leq 1$,
- $(\nabla_P)_2$ p_0 will be an endpoint of L^*,
- $(\nabla_P)_3$ $h(p_0) = l_-$,
- $(\nabla_P)_4$ $\mathrm{tp}^{p,P}(h) = t^-$, in particular, $h \restriction P$ is surjective,
- $(\nabla_P)_5$ $h(p_{|P|-1}) = l^-_{|L|-1}$, is an endpoint of L,

(∇_Q) $Q = \{q_0, q_1, \ldots, q_{|L|-1}\}$,
- $(\nabla_Q)_1$ q_i is connected with q_j iff $|i-j| \leq 1$,
- $(\nabla_Q)_2$ q_0 will be an endpoint of L^*,
- $(\nabla_Q)_3$ $h \restriction Q : Q \to L$ is an isomorphism (so $h(q_0)$ is an endpoint of L),
- $(\nabla_Q)_4$ $h(q_{|L|-1}) = l^+_{|L|-1}$ is the other endpoint of L,

(∇_S) $S = \{s_0, s_1, \ldots, s_{|S|-1}\}$ is a linear graph
- $(\nabla_S)_1$ s_i, s_j is connected iff $|i-j| \leq 1$,
- $(\nabla_S)_2$ s_0 is connected with $p_{|P|-1}$, and $h(s_0) = h(p_{|P|-1})$,
- $(\nabla_S)_3$ $s_{|S|-1}$ is connected with $q_{|L|-1}$, and $h(s_{|S|-1}) = h(q_{|L|-1})$,
- $(\nabla_S)_4$ $h \restriction S : S \to L$ is surjective,
- $(\nabla_S)_5$ if $h(p_{|P|-1}) = h(q_{|Q|-1})$, then $|S| = 2|L| - 1$,
- $(\nabla_S)_6$ if $h(p_{|P|-1}) \neq h(q_{|Q|-1})$ (so they are necessarily opposite endpoints of L), then S is the linear graph with $|S| = |L|$,

Note that $h(s_{|S|-1})$, $h(s_0)$ are endpoints, and so it follows from the surjectivity of $h \restriction S$ ($(\nabla_S)_4$) together with $(\nabla_S)_5$, $(\nabla_S)_6$ that $h(s_{|S|-1})$, $h(s_0)$ fully determine $h \restriction S$.

We construct P as follows. The linear graph P will be the disjoint union of the intervals $P_0, P_1, \ldots, P_{k-1}$ for some k (where P_i is connected with P_{i-1}, and P_{i+1}).

We recall that $l_0 = l_-$, and $\{l^-_i : i < k\}$ ($k = 1, 2, \ldots, |L|$) is an increasing chain of connected subgraphs (intervals) of L, coding the type t^-. We define the sequence $i_0, i_1, \ldots i_{j_0}$ (as well as j_0) by recursion as follows. First, set $i_0 = 0$, and assuming that i_j is defined we let i_{j+1} be minimal such that $i_{j+1} > i_j$ and $\neg(l^-_{i_{j+1}-1} R l^-_{i_{j+1}})$, if such i_{j+1} exists. If this cannot be continued, and we have defined $i_0, i_1, \ldots, i_j$, then define $i_{j+1} = |L|$, and let $j_0 = j+1$), and we stipulate that P will be the disjoint union of P_j's, where $j < j_0$. (Note that $j_0 \geq 2$, otherwise we are in the case of Lemma B.2).

Let $P_0 = \{p_0, p_1, \ldots, p_{i_1-1}\}$ be a copy of $\{l^-_0, l^-_1, \ldots, l^-_{i_1-1}\}$, so it has endpoints $l^-_0 = l_-$, and $l^-_{i_1-1}$, and define $h(p_j) = l^-_j$ for $j < i_1$, i.e.

$(\bigstar)_0$ $h \restriction P_0$ is one-to-one, maps P_0 onto the interval between $l^-_0 = h(p_0)$ and $l^-_{i_1-1} = h(p_{i_1-1})$, and for future reference we note the following:

$$h[P_0] = \{l^-_0, l^-_1, \ldots, l^-_{i_1-1}\}.$$

For $j < j_0$ we let $P_j = \{p_{|P_{<j}|}, p_{|P_{<j}|+1}, \ldots, p_{|P_{<j}|+|P_j|-1}\}$ (where $P_{<j}$ stands for $\bigcup_{k<j} P_k$), and define $h \restriction P_j$ in such a way that for each $j < j_0$ (with $j \geq 1$) the following hold.

$(\bigstar)_j$ $h \restriction (P_j \cup \{p_{|P_{<j}|-1}\}$ is one-to-one, mapping $P_j \cup \{p_{|P_{<j}|-1}\}$ onto the interval between $l^-_{i_j-1} = h(p_{|P_{<j}|-1})$ and $l^-_{i_{j+1}-1} = h(p_{|P_{<j}|+|P_j|-1})$, and this interval

can be written as

$$h[P_j \cup (\{p_{|P_{<j}|-1}\}] = \{l_0^-, l_1^-, \dots, l_{i_{j+1}-1}^-\}.$$

Note that we defined i_{j_0} to be $|L|$, so

$(\bigstar)_{j_0}$ $l_{i_{j_0}-1}^- = l_{|L|-1}^- = h(p_{|P|-1})$ is necessarily an endpoint of L.

Moreover, $l_{(i_{j_0}-1)-1}^-$ must be the other endpoint of L. It is not difficult to see that $\mathrm{tp}^{p_0,P}(h) = t^-$. This finishes the construction of P and $h \restriction P$.

It is not difficult to see that there is a unique Q and $h\restriction Q$ satisfying $(\nabla_Q)_1$-$(\nabla_Q)_4$.

So by the fact that $h(q_{|L|-1}) = l_{|L|-1}^+$ is an endpoint of L ($(\nabla_Q)_4$) and $(\bigstar)_{j_0}$ the interval S, lying in between P and Q can be defined as required in, $(\nabla_S)_4$, $(\nabla_S)_5$, $(\nabla_S)_6$, i.e.

$(\bigstar)_S$ if $h(s_0) = h(p_{|P|-1})$ and $h(q_{|L|-1}) = h(s_{|S|-1})$ are opposite endpoints of L, then $|S| = L$, and $h\restriction S$ is a bijection, or else, if $h(p_{|P|-1}) = h(q_{|L|-1}) = h(s_{|S|-1})$, then $|S| = 2|L| - 1$ and

$$h\restriction\{s_0, s_1, \dots, s_{|L|-1}\} \text{ and } h\restriction\{s_{|L|-1}, s_{|L|}, \dots, s_{2|L|-1}\}$$

are both bijections with L.

Now, having constructed $L^* = P \cup S \cup Q$ and h, fix M, $f \in \mathrm{Epi}(M, L)$, we need to construct M^*, $g \in \mathrm{Epi}(M^*, M)$ and to lift up f to some $f^* : M^* \to L^*$ with $h \circ f^* = f \circ g$.

M^* will be defined to be the disjoint union of the subgraphs U, V and W, where $f^*[U] = P$, $f^*[V] = Q$, $f^*[W] = S$. We begin with constructing U, and $f^*\restriction U$. Using j_0 from the construction of P, and recalling that $\mathrm{tp}^{m_+,M}(f) = t^+$ we define the sequence $k_0, k_1, \dots, k_{j_0}$ as follows. First we fix

$$\text{(B.3)} \qquad \begin{array}{l} \text{an enumeration } \{m_0, m_1, \dots, m_{|M|-1}\} \text{ of } M \\ \text{with } m_0 = m_+,\ m_{|M|-1} = m_- \text{ and } m_j R m_{j+1}. \end{array}$$

Let $k_0 = 0$, and let k_j be minimal k such that $f(m_k) = l_{i_j-1}^-$. Note that $\mathrm{tp}^{m_+,M}(f) = t^-$ implies that $k_j < k_{j+1}$. Now we note that f maps the interval $M_0 := \{m_0 = m_{k_0}, m_1, \dots, m_{k_1}\}$ onto the interval between $f(m_0) = l_0^-$ and $f(m_{k_1}) = l_{i_1-1}^-$, and similarly h maps P_0 onto this interval, endpoints p_0 and $p_{l_{i_1-1}}$ being mapped to l_0^- and $l_{i_1-1}^-$ by $(\bigstar)_j$. Therefore, we can invoke Lemma B.2, and obtain

$(\ast)_0$ $U_0 = \{u_0, \dots, u_{|U_0|-1}\}$, and $f^*\restriction U_0 \in \mathrm{Epi}(U_0, P_0)$, $g\restriction U_0 \in \mathrm{Epi}(U_0, M_0)$ with
- $h \circ f^*\restriction U_0 = f \circ g\restriction U_0$,
- $f^*(u_0) = p_0$,
- $f^*(u_{|U_0|-1}) = p_{|P_0|-1} (= p_{i_0-1})$,
- $g(u_0) = m_0 (= m)$,
- $g(u_{|U_0|-1}) = m_{k_1}$.

If $0 < j < j_0$, then we note that f maps the interval $M_j := \{m_{k_j}, m_{k_j+1}, \dots, m_{k_{j+1}}\}$ onto an interval containing $f(m_{k_j}) = l_{i_j-1}^-$ and $f(m_{k_{j+1}}) = l_{i_{j+1}-1}^-$, moreover, this interval is a subset of $\{l_0^-, l_1^-, \dots, l_{i_{j+1}-1}^-\}$. On the other hand, $P_j \cup \{p_{|P_{<j}|-1}\}$ has endpoints $p_{|P_{<j}|-1}$, $p_{|P_{<j}|+|P_j|-1}$, and is mapped by h onto the interval between

$l^-_{i_j-1} = h(p_{|P_{<j}|-1})$ and $l^-_{i_{j+1}-1} = h(p_{|P_{<j}|+|P_j|-1})$ by $(\bigstar)_j$. But $(\bigstar)_j$ also says that $l^-_{i_j-1}$ $l^-_{i_{j+1}-1}$ are endpoints of the interval $\{l^-_0, l^-_1, \dots, l^-_{i_{j+1}-1}\}$. So by the above,

$$f[M_j] = h[P_j \cup \{p_{|P_{<j}|-1}\}] = \{l^-_0, l^-_1, \dots, l^-_{i_{j+1}-1}\},$$

and both f and h map endpoints to endpoints. So again by Lemma B.2 we define

$(\ast)_j$ $U_j = \{u_{|U_{<j}|}, \dots, u_{|U_{<j}|+|U_j|-1}\}$, and $f^*\restriction U_j \in \mathrm{Epi}(U_j, P_j \cup \{p_{|P_{<j}|-1}\})$, $g\restriction U_j \in \mathrm{Epi}(U_j, M_j)$ with
- $f^*(u_{|U_{<j}|}) = p_{|P_{<j}|-1}$,
- $f^*(u_{|U_{<j}|+|U_j|-1}) = p_{|P_{<j}|+|P_j|-1}$,
- $g(u_{|U_{<j}|}) = m_{k_j}$,
- $g(u_{|U_{<j}|+|U_j|-1}) = m_{k_{j+1}}$.

This way

$(\ast)_U$ we defined
- $U = \bigcup_{j<j_0} U_j$ (as a disjoint union, with U_j connected to U_{j+1}),
- the epimorphism $f^*\restriction U$, which maps U onto P, with $f^*(u_0) = p_0$, $f^*(u_{|U|-1}) = p_{|P|-1}$,
- the epimorphism $g\restriction U$ mapping U to the interval $\{m_0, m_1, \dots, m_{k_{j_0}}\}$ in M, with $g(u_0) = m_0$, $g(u_{|U|-1}) = m_{k_{j_0}}$.

Moreover, $h \circ f^*\restriction U = f \circ g\restriction U$, so (by $(\bigstar)_{j_0}$)

$$f(g(u_{|U|-1}) = h(f^*(u_{|U|-1}) = h(p_{|P|-1}) = l^-_{|L|-1} (= l^-_{i_{j_0}-1}). \tag{B.4}$$

$(\ast)_V$ We construct $V = \{v_0, v_1, \dots, v_{|V|-1}\}$,
-
- $g\restriction V$ is a bijection between V and a subinterval of M with $g(v_0) = m_-$ (which is $m_{|M|-1}$ according to our labeling from (B.3)), and $m_{j'}$, where j' is maximal such that

$$f\restriction\{m_{j'}, m_{j'+1}, \dots, m_{|M|-1}\} \text{ maps onto } L,$$

  equivalently, $f(m_{j'}) = l^+_{|L|-1}$,
- $f^*\restriction V \in \mathrm{Epi}(V, Q)$ such that $h \circ f^*\restriction V = f \circ g\restriction V$, so mapping V onto Q with $h(f^*(v_0)) = l^+_0$, $h(f^*(v_{|V|-1})) = l^+_{|L|-1}$, so necessarily $f^*(v_{|V|-1}) = q_{|L|-1}$ (which can be done in a unique way, as $h\restriction Q$ is a bijection).

Moreover, the fact $h \circ f^*\restriction V = f \circ g\restriction V$ (and $(\nabla_Q)_4$) implies

$$f(g(v_{|v|-1}) = h(f^*(v_{|V|-1}) = h(q_{|L|-1}) = l^+_{|L|-1}. \tag{B.5}$$

It remains to define the linear graph W whose endpoints are connected with $u_{U|-1}$ and $v_{|V|-1}$, respectively, and define $f^*\restriction W : W \to S$ and $g\restriction W$.

W will be of the form

$(\ast)_W$ $W = \{w_0, w_1, \dots, w_{|W|-1}\}$ with $w_i R w_{i+1}$,
- and its endpoint w_0 connected with $u_{|U|-1}$, while the other endpoint $w_{|W|-1}$ is connected with $v_{|V|-1}$,
- and we let $f^*(w_0) = s_0$, $g(w_0) = g(u_{|U|-1})$,
- moreover let $f^*(w_{|W|-1}) = s_{|S|-1}$, $g(w_{|W|-1}) = g(v_{|V|-1})$.

It is easily checked (e.g. by (B.4), (B.5), $(\bigstar)_S$) that $f \circ g$ agrees with $h \circ f^*$ on the endpoints of W i.e. on w_0, $w_{|W|-1}$, and the attained values are endpoints of L:

$$f(g(w_0)) = h(f^*(w_0) = l^-_{|L|-1},$$

$$f(g(w_{|W|-1})) = h(f^*(w_{|W|-1}) = l^+_{|L|-1}.$$

If these two are opposite endpoints, then we can apply Lemma B.2 (to L, $h \restriction S \in \mathrm{Epi}(S, L)$ and the subinterval of M between $g(u_{|U|-1}) = g(w_0)$ and $g(v_{|V|-1}) = g(w_{|W|-1})$) and so we obtain W, $f^* \restriction W$, $g \restriction W$.

If this is not the case, then necessarily $f(g(w_0)) = f(g(w_{|W|-1}))$, and then pick $m' \in M$ with $f(m')$ being the other endpoint of L. This case we split S into two, $S = S_0 \cup S_1$, where $S_0 = \{s_0, s_1, \dots, s_{|L|-1}\}$, $S_1 = \{s_{|L|-1}, s_{|L|}, \dots, s_{2|L|-1}\}$. as well as find W in the form of $W_0 \cup W_1$. Note that this case $h(s_0) = h(s_{|S|-1})$, so (by $(\nabla_S)_4$, $(\nabla_S)_5$) we have $f(m') = h(s_{|L|-1})$. We apply Lemma B.2 twice:

- to L, $h \restriction S_0 \in \mathrm{Epi}(S_0, L)$ and f restricted to the interval in M between $g(u_{|U|-1}) = g(w_0)$ and m' (to obtain W_0 and $g \restriction W_0$, $f^* \restriction W_0$)
- and to L, $h \restriction S_1 \in \mathrm{Epi}(S_1, L)$ and f restricted to the interval between m' and $g(v_{|V|-1}) = g(w_{|W|-1})$, with which we get W_1, $g \restriction W_1$, $f^* \restriction W_1$.

It is straightforward to check that $M^* = U \cup V \cup W$, g, f^* are as required (where the endpoints of M^* are $m^*_+ = u_0$ and $m^*_- = v_0$). $\square_{\mathrm{Lemma B.3}}$

We are ready to prove Lemma B.1.

Proof of Lemma B.1. It is straightforward to check that Lemma B.3 together with Lemma B.2 imply the main part of Lemma B.1.

We argue that the moreover clause follows from the main part of the lemma. Pick $m_+ \in M$, $m'_+ \in M'$ such that $f(m_+) = f'(m'_+)$ is an endpoint of L. Let $\alpha : N \to M$ be an epimorphism which maps the endpoint n_- (n_+, resp.) of N to m_- (m_+, resp.) and similarly, $\alpha' : N' \to M'$ is such that n'_- goes to m'_-, and n'_+ is sent to m'_+. Since

$$f \circ \alpha(n_+) = f(m_+) = f'(m'_+) = f \circ \alpha(n'_+) \text{ is an endpoint of } L,$$

clearly

$$\mathrm{tp}^{n_+, N}(f \circ \alpha) = \mathrm{tp}^{n'_+, N'}(f' \circ \alpha'). \tag{B.6}$$

On the other hand, $\alpha(n_-) = m_-$ is an endpoint, so necessarily

$$\mathrm{tp}^{n_-, N}(f \circ \alpha) = \mathrm{tp}^{m_-, M}(f),$$

and similarly

$$\mathrm{tp}^{n'_-, N'}(f' \circ \alpha) = \mathrm{tp}^{m'_-, M'}(f'),$$

so by our assumptions

$$\mathrm{tp}^{n_-, N}(f \circ \alpha) = \mathrm{tp}^{n'_-, N'}(f' \circ \alpha), \tag{B.7}$$

Using (B.6), (B.7) we can apply the lemma to $f \circ \alpha$ and $f' \circ \alpha'$. Replacing the resulting g and g' by $\alpha \circ g$ and $\alpha' \circ g'$, respectively, we are done. $\square_{\mathrm{Lemma B.1}}$

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