CLOSED REGULARITY OF HAAR POSITIVE SETS AND THE EXACT COMPLEXITY OF THE HAAR NULL IDEAL

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Abstract. We show that every non-Haar null analytic subset of $\mathbb{Z}^\omega$ contains a non-Haar null closed subset. We also extend this to higher projective classes assuming determinacy. Then we strengthen a result of Solecki and answer a question of Matheron-Zelený by determining the exact descriptive complexity of the Haar null ideal, that is, we prove that the set of closed Haar null subsets of $\mathbb{Z}^\omega$ in the Effros Borel space is $\mathbf{\Sigma}^1_1$-inductive Borel-complete. We also lift this result from closed Haar null sets to $\mathbf{\Sigma}^1_1$ Haar null sets. As a by-product, we extend a result of Saint-Raymond from closed sets to $\mathbf{\Sigma}^1_1$ sets.

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Introduction

It is not hard to see that non-locally compact Polish groups do not admit Haar measures (that is, invariant, regular, σ-finite Borel measures). However, Christensen [4] (and later, independently, Hunt-Sauer-Yorke [9]) generalized the ideal of Haar measure zero sets to every Polish group as follows:

Definition 0.1. Let $(G, \cdot)$ be a Polish group. We say that $A \subset G$ is *Haar null*, (in symbols, $A \in \mathcal{HN}$) if there exists a universally measurable set $U \supset A$ (that is, a set measurable with respect to the completion of every Borel probability measure) and a Borel probability measure μ on G such that for every $g, h \in G$ we have $\mu(gUh) = 0$. Such a measure μ is called a *witness measure* for A.

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It is well-known that these sets are invariant under subsets, countable unions and two-sided translations. Moreover, they are indeed a well-behaved generalization of sets of Haar measure zero, since in the locally compact case Haar null sets and Haar measure zero sets coincide.

This notion has found wide application in diverse areas such as functional analysis, dynamical systems, group theory, geometric measure theory, and analysis (see, e.g., [15, 2, 18, 17, 5, 1]). It provides a well-behaved notion of "almost every" (or "prevalent") element of a Polish group. It is therefore very natural to investigate the regularity properties of Haar null sets. In particular, one might wonder whether "small sets are contained in nice small sets" and whether "large sets contain nice large sets". Concerning the first question, Solecki [17] has shown a positive statement, namely, that every analytic Haar null set is contained in a Borel Haar null set. On the negative side, the first and the third authors [6] proved that, unlike the situation in locally-compact groups, in non-locally compact abelian Polish groups there are Borel Haar null sets that have no G_δ Haar null supersets.

The first main goal of the present paper is to answer positively the second question in case of perhaps the most important non-locally compact Polish group, $\mathbb{Z}^\omega$ (often called the Baer-Specker group), that is, the ω'th power of the additive group of the integers.

Theorem 0.2. *Every analytic non-Haar null subset of $\mathbb{Z}^\omega$ contains a closed non-Haar null subset. Moreover, assuming $\mathbf{\Pi^1_n}$-determinacy, every $\mathbf{\Sigma^1_{n+1}}$ non-Haar null set in $\mathbb{Z}^\omega$ admits a closed non-Haar null subset.*

Our proof is based on results of Solecki [17] and Brendle-Hjorth-Spinas [3]. Roughly speaking, a theorem from the former paper allows us to use witness measures of a very special form, and thus to reduce the understanding of the Haar null ideal to the understanding of the non-dominating ideal (see Section 1 for the definitions), while the latter contains the regularity properties of the latter ideal. The reduction is based on a coding map and utilizes a compactness argument.

Note that there is a common alternative definition of Haar null sets where one requires the superset U to be Borel instead of universally measurable. By the above result of Solecki, this would not affect the first statement of Theorem 0.2, but, curiously, with this definition the second statement concerning projective sets would simply fail: it is proved in [6] that there is a coanalytic set that is Haar null with the definition of the present paper but not with the Borel supersets, and it is easy to see that this set cannot contain non-Haar null closed sets.

As the second main goal of our paper, we utilize the same methods to calculate the exact descriptive complexity of the set $\{C \in \mathcal{F}(\mathbb{Z}^\omega) : C \in \mathcal{HN}\}$, that is, the class of Haar null sets in the Effros space. There has already been several attempts at this problem, first Solecki [17] showed that this set is $\mathbf{\Sigma^1_2}$, but neither $\mathbf{\Sigma^1_1}$ nor $\mathbf{\Pi^1_1}$. Then Matheron and Zelený [12] proved that it is so called $\mathbf{\Sigma^1_1}$-inductive Borel-hard (see the definition in Section 1), and also asked what the exact complexity is. Here we provide a final answer by proving that this set is actually $\mathbf{\Sigma^1_1}$-inductive Borel-complete. This class of $\mathbf{\Sigma^1_1}$-inductive sets is an interesting subclass of $\mathbf{\Delta^1_2}$ (in particular, we obtain that the above set is also $\mathbf{\Pi^1_2}$, which had not been known before), and we note that a key difference between $\mathbf{\Sigma^1_1}$-inductive and $\mathbf{\Delta^1_2}$ is that $\mathbf{\Sigma^1_1}$-inductive sets are always measurable and have the Baire property. See [13, 12, 10] for more details on this class of sets. Finally, we address the question of the complexity of the Haar null ideal among Borel or even analytic sets instead of closed ones. The answer turns out to be $\mathbf{\Sigma^1_1}$-inductive complete. (Note that here we have the formally stronger completeness instead of Borel-completeness.)

In the course of the proof of this last theorem, as a by-product, we extend a result of Saint-Raymond from closed sets to $\mathbf{\Sigma^1_1}$ sets; he proved that [16, Theorem 11.] the closed sets that are not cofinal wrt. $\leq$ form a $\mathbf{\Sigma^1_1}$-inductive Borel-complete set, and strengthen this to analytic sets and completeness, see Theorem 3.4 and Theorem 4.7.

1. Preliminaries and basic facts

We start with the most important definitions and theorems that we will need. We will adapt the notation from [11] for descriptive set theoretic concepts.

Definition 1.1. A set $A \subset G$ in a Polish group is called *compact-catcher*, if for every compact set $K \subset G$ there are $g, h \in G$ such that $gKh \subset A$. The set A is *compact-biter*, if for every non-empty compact set $K \subset G$ there are $g, h \in G$ and a non-empty relatively open subset $U \subset K$ such that $gUh \subset A$.

Fact 1.2. *It is easy to check that compact-catcher and compact-biter sets are not Haar null.*

Definition 1.3. Let J be a countably infinite set. Then let $J^{<\omega}$ denote the set of finite sequences with elements from J. A *tree on J* is a subset of $J^{<\omega}$ that is closed under taking initial segments. The set of trees on J is denoted by $\mathcal{T}r(J)$. Since $J^{<\omega}$ is countable, the power set if it carries a natural compact topology homeomorphic to $\mathcal{P}(\omega)$ and hence to the Cantor space. Since $\mathcal{T}r(J)$ is a closed subset of this space, it also carries a natural compact Polish topology. A tree is *pruned* if every element of it has a proper extension. The set of pruned trees on J is denoted by $\mathcal{P}r\mathcal{T}r(J)$. We equip it with the subspace topology inherited from the topology of $\mathcal{P}(J^{<\omega})$.

Lemma 1.4. *$\mathcal{P}r\mathcal{T}r(J)$ is homeomorphic to ω^ω.*

Proof. By [11, Theorem 7.7.], it suffices to check that $\mathcal{P}r\mathcal{T}r(J)$ is non-empty, zero-dimensional, Polish, and all compact subsets of $\mathcal{P}r\mathcal{T}r(J)$ have empty interior in $\mathcal{P}r\mathcal{T}r(J)$. Non-empty is clear, zero-dimensional follows from the fact that $\mathcal{P}(J^{<\omega})$ is zero-dimensional, and Polish can easily be checked by showing that it is a G_δ subspace. To prove the last statement, inside every basic clopen set one can construct a sequence of pruned trees converging to a tree that is not pruned (e.g. finite). $\square$

The following fact is just a trivial consequence of standard results.

Fact 1.5. *Assume that X, Y are Polish spaces, $F \subset X \times Y$ is Borel, μ is a Borel probability measure on X, and $K_0 \subset \operatorname{proj}_X(F)$ is a compact set with $\mu(K_0) > 0$. Then there exists a compact set $K \subset F$ such that $\operatorname{proj}_X(K) \subset K_0$ and $\mu(\operatorname{proj}_X(K)) > 0$.*

Proof. Using the Jankov, von-Neumann Uniformization theorem (see, [11, Theorem 18.1]) there exists a measurable function $h : K_0 \to Y$ with $\operatorname{graph}(h) \subset F$. Consequently, by Lusin's theorem, there exists a compact set $K_1 \subset K_0$ with $\mu(K_1) > 0$ and such that $h \restriction K_1$ is continuous. But then $K = \operatorname{graph}(h \restriction K_1)$ satisfies the required properties. $\square$

Fact 1.6. *Assume that X, Y are Polish spaces, $n \in \omega$, $H \subset X \times Y$ is $\mathbf{\Pi^1_n}$, μ is a Borel probability measure on X, and $\operatorname{proj}_X(H)$ is not of μ-measure zero. Then, $\mathbf{\Pi^1_n}$-determinacy implies that there exists a compact set $K \subset H$ such that $\mu(\operatorname{proj}_X(K)) > 0$. In particular, $\mathbf{\Pi^1_n}$-determinacy implies $\mathbf{\Sigma^1_{n+1}}$-sets are universally measurable, and the case $n = 0$ holds in **ZFC**.*

Proof. This is proved in [11, 36.20] for $n = 1$ assuming $\mathbf{\Sigma^1_1}$-determinacy, the proof is essentially the same. □

For $b \in \omega^\omega$ let us denote by μ_b the product of the probability measures on $\mathbb{Z}$ supported uniformly on the $[0, b(n)]$'s, hence $\operatorname{supp} \mu_b = \prod_{n \in \omega}[0, b(n)] \subset \mathbb{Z}^\omega$. For a Polish space X we will denote by $\mathcal{K}(X)$ and $\mathcal{F}(X)$ the space of non-empty compact subsets of X with the Hausdorff metric and the space of non-empty closed subsets of X with the Effros Borel structure, respectively. The following, easy to prove statements will be used:

Fact 1.7. *Let X be a Polish space and $F \subset X$ be closed. Then*

(1) *the map $\omega^\omega \to \mathcal{P}(\mathbb{Z}^\omega)$ (that is, the Polish space of the Borel probability measures on $\mathbb{Z}^\omega$) defined by $b \mapsto \mu_b$*
(2) *the map $\omega^\omega \times \mathcal{K}(\mathbb{Z}^\omega) \to \mathbb{R}$ defined by $(b, K) \mapsto \mu_b(K)$*
(3) *the map $\mathcal{K}(X \times \omega^\omega) \to \mathcal{K}(X)$ defined by $K \mapsto \operatorname{proj}_X(K)$*
(4) *the set $\{K \in \mathcal{K}(X) : K \subset F\}$*

are Borel.
Moreover, if $n \geq 1$, $H \subseteq X$ is $\mathbf{\Pi^1_n}$, then

(5) *the set $\{K \in \mathcal{K}(X) : K \subset H\}$ is $\mathbf{\Pi^1_n}$.*

Notation 1.8. *For $f, g \in \omega^\omega$ we will write $f \leq^* g$ if $f(n) \leq g(n)$ holds for each $n \in \omega$ with finitely many exceptions. Recall that a set $S \subset \omega^\omega$ is* dominating *if it is cofinal wrt. $\leq^*$. The σ-ideal of non-dominating sets is denoted by $\mathcal{ND}$.*

A connection between $\mathcal{ND}$ and $\mathcal{HN}$ has already been established by Solecki [17]. We will use the following:

Lemma 1.9. *Let $S \subset \mathbb{Z}^\omega$ be a Haar null set. There exists a $b \in \omega^\omega$ such that for every $b' \geq^* b$ we have that $\mu_{b'}$ is a witness for $S \in \mathcal{HN}$.*

Let us remark first that this statement has been implicitly proved in [17] and used without proof in [2]. We will indicate how to show it using a slightly different argument from [14].

Sketch of the proof. It is not hard to see that in order to establish the lemma it suffices to produce a b such that for every $b' \geq b$ the measure $\mu_{b'}$ is a witness for $S \in \mathcal{HN}$. Now, one can check that in the proof of [14, Theorem 3.1] only lower bounds are imposed on the sequence $(N(n))_{n\in\omega}$ and consequently on the sequence $(a(n))_{n\in\omega}$ as well. Applying this observation and [14, Theorem 3.1] for a Borel Haar null $B \supset S$, the choice $b = (a(n))_{n\in\omega}$ yields the lemma. □

Next we turn to the definitions of $\mathbf{\Sigma^1_1}$-inductive sets.

Definition 1.10. For a set $H \subset \omega^\omega$ let $G(H)$ denote the game when two players take turns and play natural numbers, hence together build a real $x \in \omega^\omega$, and Player I wins the run of the game iff $x \in H$.

Definition 1.11. For a set X and a subset $B \subset \omega^\omega \times X$ the *game quantifier* is defined as

$$\Game B = \{x \in X : \text{Player I has a winning strategy in } G(B^x)\}.$$

Definition 1.12. A set $A \subset X$ in a Polish space is $\mathbf{\Sigma^1_1}$-inductive if there exists an F_σ set $B \subset \omega^\omega \times X$ such that

$$A = \Game B.$$

This terminology will be justified by the following.

Definition 1.13. An *induction* is a map $\Phi : \mathcal{P}(\omega) \times X \to \mathcal{P}(\omega)$ such that $H \subset H'$ implies $\Phi(H, x) \subset \Phi(H', x)$ for every fixed x.

Let Φ be an induction and fix x. Then let us iterate the map $H \mapsto \Phi(H, x)$ starting with $\emptyset$:

$$\Phi^{(0)}(x) = \emptyset,$$
$$\Phi^{(\alpha+1)}(x) = \Phi(\Phi^{(\alpha)}(x), x),$$
$$\Phi^{(\alpha)}(x) = \cup_{\beta<\alpha}\Phi^{(\beta)}(x) \text{ for limit } \alpha.$$

Since this is an increasing sequence, and ω is countable, there is a least countable ordinal α such that

$$\Phi^{(\alpha)}(x) = \Phi^{(\alpha+1)}(x).$$

It is easy to see that this $\Phi^{(\alpha)}(x)$ is the least fixed point of the map $H \mapsto \Phi(H, x)$. For this α define

Definition 1.14. $\Phi^{(\infty)}(x) = \Phi^{(\alpha)}(x)$.

Definition 1.15. We say that Φ is a $\mathbf{\Sigma^1_1}$-*induction* if $\{(H, x) \in \mathcal{P}(\omega) \times X : n \in \Phi(H, x)\}$ is $\mathbf{\Sigma^1_1}$ for every n.

And finally, by results of Solovay, Wolfe and others (see [16]) we have

Theorem 1.16. *Let X be Polish and $A \subset X$. Then A is $\mathbf{\Sigma^1_1}$-inductive iff there exists a $\mathbf{\Sigma^1_1}$-induction Φ such that*

$$A = \{x \in X : 0 \in \Phi^{(\infty)}(x)\}.$$

We will also need some basic properties of $\mathbf{\Sigma^1_1}$-inductive sets.

Lemma 1.17. *Let $X \subset Y$ be Polish spaces. If $A \subset X$ is $\mathbf{\Sigma^1_1}$-inductive in X then it is also $\mathbf{\Sigma^1_1}$-inductive in Y.*

Proof. Let $\Phi : \mathcal{P}(\omega) \times X \to \mathcal{P}(\omega)$ be a $\mathbf{\Sigma^1_1}$-induction such that

$$A = \{x \in X : 0 \in \Phi^{(\infty)}(x)\}.$$

For $(H, y) \in \mathcal{P}(\omega) \times Y$ define

$$\Psi(H, y) = \begin{cases} \Phi(H, y) & \text{if } y \in X, \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly, this is a $\mathbf{\Sigma^1_1}$-induction, and

$$A = \{y \in Y : 0 \in \Psi^{(\infty)}(y)\}.$$

$\square$

Lemma 1.18. *Let X and Y be Polish spaces, $A \subset X$ $\mathbf{\Sigma^1_1}$-inductive, and $f : Y \to X$ Borel. Then $f^{-1}(A)$ is also $\mathbf{\Sigma^1_1}$-inductive.*

Proof. Let $\Phi : \mathcal{P}(\omega) \times X \to \mathcal{P}(\omega)$ be a $\mathbf{\Sigma^1_1}$-induction such that

$$A = \{x \in X : 0 \in \Phi^{(\infty)}(x)\}.$$

For $(H, x) \in \mathcal{P}(\omega) \times X$ define

$$\Psi(H, x) = \Phi(H, f(x)).$$

It is easy to check that this is a $\mathbf{\Sigma^1_1}$-induction, $\Psi^{(\alpha)}(x) = \Phi^{(\alpha)}(f(x))$ for every α and x, hence $\Psi^{(\infty)}(x) = \Phi^{(\infty)}(f(x))$ for every x, therefore

$$f^{-1}(A) = \{x \in X : 0 \in \Psi^{(\infty)}(x)\}.$$

□

Fact 1.19. *The set $S \subseteq X$ is ∂G_δ iff $X \setminus S$ is ∂F_σ.*

Proof. Pick an G_δ set $H \subseteq X \times \omega^\omega$ witnessing that S is ∂G_δ. We only have to interchange the roles of Player I and II. Define

$$H' = \{(x, \langle n \rangle ^\frown z) : \ (x, z) \in H\},$$

and let $H'' = X \times \omega^\omega \setminus H'$. It is easy to check that H'' is an F_σ set witnessing $X \setminus S$ is ∂F_σ. The same argument works for the reverse implication. □

The next definition is key to our purposes.

Definition 1.20. A set $A \subset X$ in a Polish space is $\mathbf{\Sigma^1_1}$-*inductive hard* if for every zero-dimensional Polish space Y and every $\mathbf{\Sigma^1_1}$-inductive set $B \subset Y$ there exists a continuous function $f : Y \to X$ such that $B = f^{-1}(A)$. If it is also $\mathbf{\Sigma^1_1}$-inductive, then it is called $\mathbf{\Sigma^1_1}$-*inductive complete*.

We will also need the following variation.

Definition 1.21. A set $A \subset X$ in a standard Borel space is $\mathbf{\Sigma^1_1}$-*inductive Borel-hard* if for every standard Borel space Y and every $\mathbf{\Sigma^1_1}$-inductive set $B \subset Y$ there exists a Borel function $f : Y \to X$ such that $B = f^{-1}(A)$. If it is also $\mathbf{\Sigma^1_1}$-inductive, then it is called $\mathbf{\Sigma^1_1}$-*inductive Borel-complete*.

Since every standard Borel space can be equipped with a zero-dimensional Polish topology, if a set $A \subset X$ in a Polish space is $\mathbf{\Sigma^1_1}$-inductive hard/complete then it is also $\mathbf{\Sigma^1_1}$-inductive Borel-hard/complete. However, the following question is open.

Question 1.22. *Does $\mathbf{\Sigma^1_1}$-inductive Borel-completeness imply $\mathbf{\Sigma^1_1}$-inductive completeness for a subset of a Polish space?*

Remark 1.23. Since every uncountable standard Borel space is Borel-isomorphic to the Baire space, in the definition of $\mathbf{\Sigma^1_1}$-inductive Borel-hardness/completeness it suffices to only consider the case when Y is the Baire space.

Now we show that this is actually also the case for $\mathbf{\Sigma^1_1}$-inductive hardness/completeness. Indeed, let $B \subset Y$ be a $\mathbf{\Sigma^1_1}$-inductive set in a zero-dimensional Polish space. By [11, Theorem 7.8] there is homeomorphic embedding $\varphi : Y \to \omega^\omega$ onto a closed subspace. By Lemma 1.17 $\varphi(B)$ is $\mathbf{\Sigma^1_1}$-inductive in the Baire space, hence there exists a continuous $f : \omega^\omega \to X$ such that $\varphi(B) = f^{-1}(A)$. But then $f \circ \varphi$ works.

We will need the following game of Brendle-Hjorth-Spinas [3], which we will call the BHS game.

Definition 1.24. We fix a set $H \subseteq \omega^\omega \times \omega^\omega$. Playing the BHS game on H begins with Player I playing

- $w_{-1} \in [\omega]^{<\omega}$ and $s_0 \in \omega^{w_{-1}}$,
- $w_0 \in [\omega]^{<\omega}$ and
- $v_0 \in \omega^{<\omega}$

to which player II responds with

- a natural number k_0.

After the n'th move of player II, the two together have constructed the sequence

$$(s_0, w_0, v_0), k_0, (s_1, w_1, v_1), k_1, \dots, (s_{n-1}, w_{n-1}, v_{n-1}), k_{n-1},$$

at which point

(1) player I picks
   - $s_n \in \omega^{w_{n-1}}$,
   - $w_n \in [\omega]^{<\omega}$,
   - $v_n \in \omega^{<\omega}$,

   such that $s_n(i) \geq k_{n-1}$ for each $i \in \mathrm{dom}(s_n)$,

to which

(2) player II responds with a natural number k_n.

Player I wins iff $w = \bigcup_{i\in\omega} w_i$ belongs to ω^ω (i.e. it is a total function), $v_i \neq \emptyset$ for infinitely many i, and letting $v = (v_0 {}^\frown v_1 {}^\frown \dots)$ be the resulting real we have $(v, w) \in H$.

Fact 1.25.

(1) The BHS game played on H is determined if H is Borel.
(2) If $H \in \mathbf{\Pi}^1_n$, then $\mathbf{\Pi}^1_n$-determinacy implies that the BHS game played on H is determined.

Proof. This is a corollary of the following. □

Fact 1.26. *There exist a G_δ set $P \subseteq \omega^\omega$, and a continuous surjection $f : P \to \omega^\omega \times \omega^\omega$ such that for every $H \subseteq \omega^\omega$ Player I has a winning strategy for the BHS game played on H iff I has a winning strategy in the game $G(f^{-1}(H))$ (and similarly, Player II has a winning strategy in either game iff they have a winning strategy in both games).*

Proof. Use a bijection between the set of allowed moves (s, w, v), k and the set ω. The condition that $\bigcup_{i\in\omega} s_i$ as well as $v = (v_0 {}^\frown v_1 {}^\frown \dots)$ are total functions defined on ω are easily checked to be G_δ. □

For the sake of completeness we include a proof for the following.

Lemma 1.27. *([3, Theorem 1.1])*
Suppose that the BHS game played on $H \subseteq \omega^\omega \times \omega^\omega$ is determined. Then Player I has a winning strategy iff $\mathrm{proj}_1(H)$ is dominating, so Player II has a winning strategy if and only if $\mathrm{proj}_1(H) \in \mathcal{ND}$.

Proof. It is clear that a winning strategy of I implies that $\mathrm{proj}_1(H)$ is dominating.

Fix a winning strategy σ for II, so σ is a function that assigns a natural number k to sequences of the form $(s_0, w_0, v_0), k_0, (s_1, w_1, v_1), k_1, \dots, (s_{n-1}, w_{n-1}, v_{n-1})$. Note that if σ is a winning strategy, and σ' is pointwise greater than σ, then σ' is a winning strategy, too. Also, it is enough if σ is defined on sequences $(s_0, w_0, v_0), k_0, (s_1, w_1, v_1), k_1, \dots, (s_{n-1}, w_{n-1}, v_{n-1})$ where $s_i(j) \geq k_i$ for $j \in w_{i-1}$, i.e. I doesn't break the rule. Now for each sequence $(s_0, w_0, v_0), (s_1, w_1, v_1), \dots, (s_{n-1}, w_{n-1}, v_{n-1})$ there are only finitely many possible good $k_0, k_1, \dots, k_{n-1}$, so we can assume that σ is defined on sequences of the form $(s_0, w_0, v_0), (s_1, w_1, v_1), \dots, (s_{n-1}, w_{n-1}, v_{n-1})$.

Similarly, for each (s, w, v) there are only finitely many $(s_0, w_0, v_0), (s_1, w_1, v_1), \dots, (s_{n-1}, w_{n-1}, v_{n-1})$ with $s = \bigcup_{i<n} s_i$, $v = (v_0 \frown v_1 \frown \dots \frown v_{n-1})$, and $w_{n-1} = w$, by increasing $\sigma((s_0, w_0, v_0), (s_1, w_1, v_1), \dots, (s_{n-1}, w_{n-1}, v_{n-1}))$, or $\sigma(s, v, w)$ if necessary, w.l.o.g. we can assume that

$$\sigma((s_0, w_0, v_0), (s_1, w_1, v_1), \dots, (s_{n-1}, w_{n-1}, v_{n-1})) = \sigma\left(\bigcup_{i<n} s_i, w_{n-1}, (v_0 \frown \dots \frown v_{n-1})\right).$$

Finally, we can assume that $\sigma(s, w, v) < \sigma(s', w', v')$ if $s \subseteq s'$, $w \cup \mathrm{dom}(s) \subseteq w' \cup \mathrm{dom}(s')$, $v \subseteq v'$ and one of the above is proper inclusion.

Assuming that a winning strategy σ for II exists with the above properties we are going to define $y \in \omega^\omega$ and prove that for all $x \in \mathrm{proj}_1(H)$ we have $(\exists^\infty n)\ x(n) < y(n)$.

For a fixed n we define $y(n)$ by the following recursion. We are going to define the sequence $k_0^{(n)}, k_1^{(n)}, \dots, k_n^{(n)}$ be recursion (and we will let $y(n) = k_n^{(n)}$). Set $k_0^{(n)} = \sigma(\emptyset, n+1, \emptyset)$, define

$$k_{i+1}^{(n)} = \max\{\sigma(s, n+1 \setminus \mathrm{dom}(s), v) :\ s \in (k_i)^q, q \in [n]^{\leq i+1},\ v \in (k_i)^{\leq i+1}\}, \tag{1.1}$$

and let $y(n) = k_n$. It is easy to see that

$$k_0^{(n)} < k_1^{(n)} < \dots < k_n^{(n)} \tag{1.2}$$

by our assumptions.

Fix $(x, z) \in H$, $m \in \omega$, we are going to find $m' \geq m$, such that $y(m') > x(m')$. Let $n_0 > m$ be large enough so that

$$k_0^{(n_0)} = \sigma(\emptyset, n_0 + 1, \emptyset) > \max\{x(j) :\ j < m\}. \tag{1.3}$$

Claim 1.28. *Either*

(*i*) *there exists* $s_0 \subseteq x \restriction n_0$ *with* $x \restriction m \subseteq s_0$, $v_0 \subseteq z \restriction n_0$, *such that*

$$(\forall i \in n_0 + 1 \setminus \mathrm{dom}(s_0))\ x(i) \geq \sigma(s_0, n_0 \setminus \mathrm{dom}(s_0), v_0),$$

(*ii*) *or* $x(n_0) < y(n_0)$.

Proof. First we let $s_0^* = x \restriction m$, $v_0^* = \emptyset$. Define $i_1 \in n_0 + 1 \setminus m$ be such that $x(i_1) < \sigma(s_0^*, n_0 + 1 \setminus \mathrm{dom}(s_0^*), v_0^*)$ if such i_1 exists. Let s_1^* extend s_0^* with $\mathrm{dom}(s_1^*) = \mathrm{dom} s_0^* \cup \{i_1\}$, $s_1^* \subseteq x$ if i_1 is defined (otherwise let $s_1^* = s_0^*$). Second, if i_1 exists and $z(0) < \sigma(s_0^*, n_0 + 1 \setminus \mathrm{dom}(s_0^*), v_0^*)$, then let v_1^* be $z \restriction 1$, otherwise $v_1^* = v_0^*$. We can define similarly i_2, s_2^*, v_2^* as well as $i_\ell, s_\ell^*, v_\ell^*$ for $\ell \leq n_0 - m + 1$, with the rule that if s_ℓ^* freezes then we freeze v_ℓ^*, too (i.e. $s_\ell^* = s_{\ell+1}^*$ implies $v_\ell^* = v_{\ell+1}^*$).

At the end, we have $s_{n_0 - m + 1}^* \subseteq x \restriction n_0 + 1$, $v_{n_0 + 1 - m}^* \subseteq z \restriction n_0 + 1 - m$.

First we deal with the case when $n_0 = i_p$ for some $1 \leq p \leq n_0 - m + 1$, so

$$x(n_0) < \sigma(s^*_{p-1}, n_0 + 1 \setminus \mathrm{dom}(s^*_{p-1}), v^*_{p-1}). \tag{1.4}$$

Now using (1.3) and (1.1) one easily checks that

$$\sigma(x \restriction m, n_0 + 1 \setminus m, \emptyset) = \sigma(s^*_0, n_0 + 1 \setminus \mathrm{dom}(s^*_0), v^*_0) \leq k^{(n_0)}_m.$$

Then by induction, for every $\ell < p$

$$\sigma(s^*_\ell, n_0 + 1 \setminus \mathrm{dom}(s^*_\ell), v^*_\ell) \leq k^{(n_0)}_{m+\ell},$$

thus by (1.2) and (1.4)

$$x(n_0) < \sigma(s^*_{p-1}, n_0 + 1 \setminus \mathrm{dom}(s^*_{p-1}), v^*_{p-1}) \leq k^{(n_0)}_{m+p-1} \leq k^{(n_0)}_{n_0} = y(n_0).$$

Second, if n_0 is not i_p for any $1 \leq p \leq n_0 - m + 1$, then clearly $s^*_{n_0-m} = s^*_{n_0-m+1}$ (so $v^*_{n_0-m} = v^*_{n_0-m+1}$), and letting $s_0 = s^*_{n_0-m+1}$, $v_0 = v^*_{n_0-m+1}$ clearly (ii) holds. □

Lemma 1.29. *Suppose that there exists $s_0 \subseteq x \restriction n_0$ with $x \restriction m \subseteq s_0$, $v_0 \subseteq z \restriction n_0$, such that*

$$(\forall i \in n_0 + 1 \setminus \mathrm{dom}(s_0))\ x(i) \geq \sigma(s_0, n_0 + 1 \setminus \mathrm{dom}(s_0), v_0).$$

If $n_1 > n_0$ is maximal such that

$$(\forall i \in n_1 \setminus \mathrm{dom}(s_0))\ x(i) \geq \sigma(s_0, n_1 \setminus \mathrm{dom}(s_0), v_0),$$

then either

- (i) *there exists $s_1 \subseteq x \restriction n_1$, v_1 such that $\mathrm{dom}(s_1) \cap \mathrm{dom}(s_0) = \emptyset$, $v_0 ^\frown v_1 \subseteq z \restriction n_1$, and*

$$\begin{aligned}&(\forall i \in n_1 + 1 \setminus (\mathrm{dom}(s_1) \cup \mathrm{dom}(s_0)) :\\ &x(i) \geq \sigma(s_0 \cup s_1, n_1 + 1 \setminus (\mathrm{dom}(s_0) \cup \mathrm{dom}(s_1)), v_0 ^\frown v_1),\end{aligned}$$

- (ii) *or $x(n_1) < y(n_1)$.*

Proof. Let $s^*_0 = v^*_0 = \emptyset$, define i_1 to be the least $i \in n_1 + 1 \setminus (\mathrm{dom}(s^*_0) \cup \mathrm{dom}(s_0))$ such that

$$\begin{aligned} x(i) \quad < \quad &\sigma(s_0 \cup s^*_0, (n_1 + 1) \setminus (\mathrm{dom}(s^*_0) \cup \mathrm{dom}(s_0)), v_0 ^\frown v^*_0) = \\ &= \sigma(s_0, (n_1 + 1) \setminus (\mathrm{dom}(s^*_0) \cup \mathrm{dom}(s_0)), v_0)\end{aligned}$$

(which exists by our assumptions), and let $s^*_1 = x \restriction \{i_1\}$. Define v^*_1 to be $v^*_0 {}^\frown (z(|v_0| + |v^*_0|))$ if

$$z(|v_0| + |v^*_0|) = z(|v_0|) < \sigma(s_0 \cup s^*_0, (n_1 + 1) \setminus (\mathrm{dom}(s^*_0) \cup \mathrm{dom}(s_0)), v_0 ^\frown v^*_0),$$

otherwise let $v^*_1 = v^*_0$.

We can define similarly i_2, s^*_2, v^*_2 as well as $i_\ell, s^*_\ell, v^*_\ell$ for $\ell \leq n_1 + 1 - |\mathrm{dom}(s_0)|$, with the rule that once $i_{\ell+1}$ is not defined we have $s^*_{\ell+1} = s^*_\ell$, $v^*_{\ell+1} = v^*_\ell$, moreover, we always choose i_ℓ to be the least i that satisfies the required inequality.

Similarly to the claim, if n_1 does not equal i_ℓ for any ℓ, then necessarily $s^*_{n_1+1-|\mathrm{dom}(s_0)|} = s^*_{n_1-|\mathrm{dom}(s_0)|}$, $v^*_{n_1+1-|\mathrm{dom}(s_0)|} = v^*_{n_1-|\mathrm{dom}(s_0)|}$, and letting $s_1 = s^*_{n_1+1-|\mathrm{dom}(s_0)|}$, $v_1 = v^*_{n_1+1-|\mathrm{dom}(s_0)|}$, we are done

Otherwise,

$$x_{n_1} = i_\ell < \sigma(s_0 \cup s^*_{\ell-1}, (n_1 + 1) \setminus (\mathrm{dom}(s^*_{\ell-1}) \cup \mathrm{dom}(s_0)), v_0 ^\frown v^*_{\ell-1}),$$

one can check that for each $\ell \leq n_1 - |\mathrm{dom}(s_0)|$ we have

$$\sigma(s_0 \cup s^*_\ell, (n_1+1) \setminus (\mathrm{dom}(s^*_\ell) \cup \mathrm{dom}(s_0)), v_0 \frown v^*_\ell) \leq k^{(n_1)}_{|\mathrm{dom}(s_0)|+\ell} \leq k^{(n_1)}_{n_1} = y(n_1),$$

and we are done.

$\square$

By the argument in the proof of the lemma we can obtain infinite sequences $s_0, s_1, \ldots$; $v_0, v_1, \ldots$; and $n_{-1} = 0 \leq m \leq n_0 < n_1 < \ldots$, such that either

- for some ℓ, $x(n_\ell) < y(n_\ell)$,
- or the following holds.
  (1) $s_\ell \subseteq x$, $\mathrm{dom}(s_\ell) \subseteq n_\ell$,
  (2) $\mathrm{dom}(s_\ell)$ $(\ell \in \omega)$ are pairwise disjoint,
  (3) $|v_\ell| \leq n_\ell - n_{\ell-1}$,
  (4) $v_0 \frown v_1 \frown \ldots \frown v_\ell \frown \ldots \subseteq z$,
  (5) for all ℓ

$$\left(\forall i \in n_{\ell+1} \setminus \bigcup_{p\leq\ell} \mathrm{dom}(s_p)\right)\ x(i) \geq \sigma\left(\bigcup_{p\leq\ell} s_p, n_{\ell+1} \setminus \bigcup_{p\leq\ell} \mathrm{dom}(s_p), v_0 \frown \ldots \frown v_\ell\right),$$

in particular,

$$(\forall i \in \mathrm{dom}(s_{\ell+1}))\ \ x(i) \geq \sigma\left(\bigcup_{p\leq\ell} s_p, \mathrm{dom}(s_{\ell+1}), v_0 \frown \ldots \frown v_\ell\right),$$

  (6) there exists $i \in n_{\ell+1} \setminus \bigcup_{p\leq\ell} \mathrm{dom}(s_p)$

$$x(i) < \sigma\left(\bigcup_{p\leq\ell} s_p, n_{\ell+1}+1 \setminus \bigcup_{p\leq\ell} \mathrm{dom}(s_p), v_0 \frown \ldots \frown v_\ell\right),$$

and the least such i is in $\mathrm{dom}(s_{\ell+1})$. Similarly, if

$$z\left(\sum_{p\leq\ell} |v_p|\right) < \sigma\left(\bigcup_{p\leq\ell} s_p, n_{\ell+1}+1 \setminus \bigcup_{p\leq\ell} \mathrm{dom}(s_p), v_0 \frown \ldots \frown v_\ell\right),$$

then $v_{\ell+1} \neq \emptyset$.

Conditions (1)-(5) shows that I can play $(s_\ell, \mathrm{dom}(s_{\ell+1}), v_\ell)_{\ell\in\omega}$ against II if II is playing according to σ, moreover, $\bigcup_{\ell\in\omega} s_\ell \subseteq x$, $v_0 \frown v_1 \frown \ldots \subseteq z$. On the other hand (6) says that $\bigcup_{\ell\in\omega} \mathrm{dom}(s_\ell) = \omega$, $\sum_{\ell<\omega} |v_\ell| = \infty$, so

$$\left(\bigcup_{\ell\in\omega} s_\ell, v_0 \frown v_1 \frown \ldots\right) = (x, z).$$

Since $(x, z) \in H$, I wins, contradicting that σ is a winning strategy for II. $\square$

In this paper solely the group $\mathbb{Z}^\omega$ will be considered, and so we will use the additive notation for the group operation.

For an integer-valued function $f : X \to \mathbb{Z}$ the notation $|f|$ and cf will be used for the function defined by $x \mapsto |f(x)|$ and $x \mapsto cf(x)$ for $x \in X$ and $c \in \mathbb{Z}$. Also we will write $f \leq g$ if for each $x \in X$ we have $f(x) \leq g(x)$.

2. Closed regularity of Haar positive sets

In this section we prove Theorem 0.2. Let us start with an easy observation.

Lemma 2.1. *Let $f \in \omega^\omega$ be arbitrary. Then the set*

$$I(f) = \{g \in \mathbb{Z}^\omega : \exists^\infty n \in \omega \ |g(n)| \leq f(n)\}$$

is Haar null in $\mathbb{Z}^\omega$.

Proof. Let $f'(n) = 2^n(f(n)+1)$. We will show that the measure $\mu_{f'}$ witnesses that $I(f)$ is Haar null. Let $h \in \mathbb{Z}^\omega$ be arbitrary. Clearly,

$$I(f) + h = \bigcap_{k\in\omega} \bigcup_{n \geq k} \{g + h : |g(n)| \leq f(n)\},$$

and for every $n \in \omega$ we have

$$\mu_{f'}(\{g + h : |g(n)| \leq f(n)\}) \leq \frac{2f(n)}{f'(n)} \leq \frac{2}{2^n},$$

Thus, using $\sum_{n\in\omega} \frac{2}{2^n} < \infty$ and the Borel-Cantelli lemma, we get that $\mu_{f'}(I(f) + h) = 0$. □

The following fact, essentially proved in [3], will play a crucial role.

Fact 2.2. *Assume that $A \subseteq \omega^\omega$ is a dominating set and $f : A \to \omega^\omega$ is a Borel function. If either*

- *$A \in \mathbf{\Sigma^1_1}(\omega^\omega)$,*
- *or for some $n > 0$, $A \in \mathbf{\Sigma^1_{n+1}}(\omega^\omega)$ and we assume $\mathbf{\Pi^1_n}$-determinacy,*

then there exists a closed set $C \subset A$ such that $C \notin \mathcal{ND}$ and $f \restriction C$ is continuous.

Sketch of the proof. We recall the BHS game from Definition 1.24. In case one we let $n = 0$, so $A \in \mathbf{\Sigma^1_{n+1}}(\omega^\omega)$. Fix a set $F \in \mathbf{\Pi^1_n}(\omega^\omega \times \omega^\omega)$ with $\mathrm{proj}_1(F) = A$. By Fact 1.25 the BHS game on F is determined, so necessarily I has a winning strategy (Lemma 1.27).

Let $W = \{s_t, w_t, v_t : t \in \omega^{<\omega}\}$ be given by the winning strategy σ of I. (This could be defined inductively, we let $(s_\emptyset, w_\emptyset, v_\emptyset)$ be the first move of I, i.e. $\sigma(\emptyset)$, and let (s_i, w_i, v_i) be $\sigma((s_\emptyset, w_\emptyset, v_\emptyset), i)$, etc.)

So $w_t \subset \omega$ and $\mathrm{dom}(s_\emptyset) \subset \omega$ are finite, $s_\emptyset : \mathrm{dom}(s_\emptyset) \to \omega$, and for $t \neq \emptyset$ we have $s_t : w_{t\restriction lh(t)-1} \to \omega$ and for all $i \in w_{t\restriction lh(t)-1}$ $s_t(i) \geq t(lh(t) - 1)$, finally, for every $x \in \omega^\omega$ we have that $\omega = \mathrm{dom}(s_\emptyset) \cup \bigcup_n w_{x\restriction n}$, where the union is disjoint. If $T \subset \omega^{<\omega}$ is a tree, define $C_{T,W} =$

$$\{y \in \omega^\omega : s_\emptyset \subset y \text{ and } \exists x \in [T] \ \forall n \in \omega (y \restriction w_{x\restriction n} = s_{x\restriction n+1})\}.$$

Since σ is a winning strategy, $C_{\omega^{<\omega},W} \subseteq A$. It is easy to see that the map $\phi_{T,W} : [T] \to C_{T,W}$ defined by assigning to $x \in [T]$ the unique $y \in C_{T,W}$ with the property that $\forall n \in \omega (y \restriction w_{x\restriction n} = s_{x\restriction n+1})$ is continuous and open. Moreover, one can also check that the set $C_{T,W}$ is closed for every T and W.

A *Laver-tree* is a subtree T of $\omega^{<\omega}$ such that it has a stem s (that is, a maximal node with $\forall t \in T$ $(t \subset s \vee s \subset t)$), and for every $t \not\subset s$ from T the set $\{n \in \omega : t^\frown(n) \in T\}$ is infinite.

Consider now the Borel map $f \circ \phi_{\omega^{<\omega},W} : \omega^\omega \to \omega^\omega$. By [8, Example 3.7] there exists a Laver-tree $T \subset \omega^{<\omega}$ such that $f \circ \phi_{\omega^{<\omega},W} \restriction [T]$ is continuous. Clearly,

$\phi_{T,W} = \phi_{\omega^{<\omega},W} \restriction [T]$, and since this map is open, $f \restriction \phi_{T,W}([T])(= C_{T,W})$ is continuous. Thus, it is enough to check that the set $C_{T,W}$ is dominating.

Let $z \in \omega^\omega$ be arbitrary. It is not hard to define an $x \in T$ such that $\phi_{T,W}(x) \geq^* z$ inductively: indeed, for every large enough n (namely, if $n > lh(s)$, where s is the stem of T), if $x \restriction n$ is defined, we can find an m so that $m > \max\{z(i) : i \in w_{x\restriction n}\}$ and $x \restriction n^\frown (m) \in T$. Then for an x obtained this way, it follows from $\forall i \in w_{x\restriction n}$ $(m < s_{x\restriction n+1}(i))$ that whenever n is large enough and $i \in w_{x\restriction n}$ then $\phi_{T,W}(x)(i) > z(i)$. The fact that $|w_t| < \aleph_0$ implies that $\phi_{T,W}(x)$ dominates z. $\square$

Now we are ready for the proof of the main result. Our strategy will be somewhat similar to the idea of the proof of Theorem 3.3, just significantly more sophisticated. To a given $\mathbf{\Sigma^1_{n+1}}(\mathbb{Z}^\omega)$ set $A \notin \mathcal{HN}$ we will assign a $\mathbf{\Pi^1_n}(\mathbb{Z}^\omega \times \omega^\omega)$ set D that encodes the witnesses for $A \notin \mathcal{HN}$, i.e., codes for possible witness measures μ and compact sets K, and translations $t \in \mathbb{Z}^\omega$ with $\mathrm{proj}_{\mathbb{Z}^\omega}(K)+t \subset A$ and $\mu(\mathrm{proj}_{\mathbb{Z}^\omega}(K)) > 0$. The coding will be constructed so that it ensures that $\mathrm{proj}_{\mathbb{Z}^\omega} D$ is dominating. Using the results of Brendle, Hjorth, and Spinas, we will chose a dominating closed subset of D with some additional properties, and from it a non-Haar null subset of A will be reconstructed. A compactness argument will yield that this set is in fact closed.

Proof of Theorem 0.2. Let $A \in \mathbf{\Sigma^1_{n+1}}(\mathbb{Z}^\omega)$ be a non-Haar null set, and assume $\mathbf{\Pi^1_n}$-determinacy. Let $H \subseteq \mathbb{Z}^\omega \times \omega^\omega$ be such that $\mathrm{proj}_{\mathbb{Z}^\omega}(H) = A$, with $H \in \mathcal{F}(\mathbb{Z}^\omega \times \omega^\omega)$ if $n = 0$, $H \in \mathbf{\Pi^1_n}(\mathbb{Z}^\omega \times \omega^\omega)$ if $n \geq 1$. Fix a Borel bijection $\psi : 2^\omega \to \mathcal{K}(\mathbb{Z}^\omega \times \omega^\omega)$ and define a Borel partial mapping $\phi : \omega^\omega \times \mathbb{Z}^\omega \times \mathbb{Z}^\omega \to \mathcal{K}(\mathbb{Z}^\omega \times \omega^\omega)$ as follows: let $(b, t, c) \in \mathrm{dom}(\phi)$ iff the conjunction of the following holds:

(1) $c - t \in 2^\omega$.
(2) $2b \leq^* |t|$.
(3) $\mathrm{proj}_{\mathbb{Z}^\omega}(\psi(c - t)) \subset \prod_{m\in\omega}[0, b(m)]$.
(4) $\mu_b(\mathrm{proj}_{\mathbb{Z}^\omega}(\psi(c - t))) > 0$.

Let us use the notation $+^p$ for the $(\mathbb{Z}^\omega \times \omega^\omega) \times \mathbb{Z}^\omega \to \mathbb{Z}^\omega \times \omega^\omega$ mapping that is the translation of the first coordinate, i.e., $(r, x) +^p t = (r + t, x)$. Define ϕ for $(b, t, c) \in \mathrm{dom}(\phi)$ by letting

$$\phi(b, t, c) = \psi(c - t) +^p t,$$

in other words, $\phi(b, t, c) = \psi(c - t) +^p t$ is the compact subset of $\mathbb{Z}^\omega \times \omega^\omega$ defined by $\{(r, x) +^p t : (r, x) \in \psi(c - t)\}$.

Finally, we will need a homeomorphism $\mathrm{bij} : \omega^\omega \to \omega^\omega \times \mathbb{Z}^\omega \times \mathbb{Z}^\omega$. In order to be precise, let us fix a concrete one by letting for every $m \in \omega$

- $\mathrm{bij}(f)(0)(m) = f(3m)$,
- for $i \in \{1, 2\}$ define $\mathrm{bij}(f)(i)(m) =$

$$\begin{cases} f(3m + i)/2, & \text{if } f(3m + i) \text{ is even,} \\ -(f(3m + i) + 1)/2, & \text{if } f(3m + i) \text{ is odd.} \end{cases}$$

Lemma 2.3. *Let $D = \{f \in \omega^\omega : \phi(\mathrm{bij}(f)) \subset H\}$. Then D is dominating.*

Proof. Assume otherwise, and let $f \in \omega^\omega$ witness this fact. Without loss of generality, we can assume that f is constant on the sets of the form $\{3m, 3m+1, 3m+2\}$, and it attains only positive values. Define an element $f' \in \mathbb{Z}^\omega$ by $f'(m) = 2(f(3m) + 1)$.

Using Lemma 2.1 and $A \notin \mathcal{HN}$ we get that $A \setminus I(3f') \notin \mathcal{HN}$. Since A is universally measurable (Fact 1.6) this yields that there exists $t \in \mathbb{Z}^\omega$ such that $(A \setminus I(3f')) - t$ is not $\mu_{f'}$-null. Now, using Fact 1.6 again for the set $H +^p (-t)$, and the measure $\mu_{f'}$ we get a compact set $K \subset H +^p (-t)$ (or, equivalently, $K +^p t \subset H$) with $\mu_{f'}(\mathrm{proj}_{\mathbb{Z}^\omega}(K)) > 0$.

Consider now the function $g = \mathrm{bij}^{-1}(f', t, t + \psi^{-1}(K))$. We claim that $g \geq^* f$ and $g \in D$, contradicting our initial assumption and thus finishing the proof. In order to see $g \geq^* f$, notice that, as $\emptyset \neq \mathrm{proj}_{\mathbb{Z}^\omega}(K) + t \subset (\prod_m [0, f'(m)] + t) \cap (A \setminus I(3f'))$, necessarily $f' + |t| \geq^* 3f'$, so $|t| \geq^* 2f'$. Then, it is straightforward to check from the definition of bij that $g \geq^* f$ holds.

Checking $g \in D$ is just tracing back the definitions: clearly, $\mathrm{bij}(g) = (f', t, t + \psi^{-1}(K)) \in \mathrm{dom}(\phi)$ holds, as we have already seen (1), (2), and for (3), (4) note that $\psi(t + \psi^{-1}(K) - t) = K$ and $\mu_{f'}(\mathrm{proj}_{\mathbb{Z}^\omega}(K)) > 0$. Finally, $\phi((f', t, t + \psi^{-1}(K))) = K +^p t \subset H$. $\square$

Using Fact 1.7 we get that $\mathrm{dom}(\phi)$ is a Borel set, and as $+^p$ is continuous, ϕ is a Borel map. Moreover, using ((4)) from Fact 1.7 if $n = 0$, or (5) if $n > 0$, $\{K \in \mathcal{K}(\mathbb{Z}^\omega \times \omega^\omega) : K \subset H\}$ is $\mathbf{\Pi^1_n}$, so D must be $\mathbf{\Pi^1_n}$ as well (so D is Borel in our main case, when $n = 0$). Since non-dominating sets form a σ-ideal, by passing to a dominating Borel subset of D, we can assume that there is an $m_0 \in \omega$ and a sequence $(\beta, \tau, \gamma) \in \omega^{m_0} \times \mathbb{Z}^{m_0} \times \mathbb{Z}^{m_0}$ such that for each $f \in D$ if $\mathrm{bij}(f) = (b, t, c)$ then $2b(k) \leq |t(k)|$ for each $k \geq m_0$ and for each $k < m_0$ we have $\beta(k) = b(k), \tau(k) = t(k), \gamma(k) = c(k)$.

Now Fact 2.2 implies the existence of a closed dominating set $C \subset D$ such that $\phi \circ \mathrm{bij} \restriction C$ is continuous. We claim that the set $C' = \mathrm{proj}_{\mathbb{Z}^\omega}(\bigcup_{x \in C} \phi(\mathrm{bij}(x)))$ is closed and non-Haar null, which finishes the proof, as it is clearly a subset of A.

First, we show that the set is closed. Let $r_m \in C'$ with $r_m \to r$ and assume that $r_m \in \mathrm{proj}_{\mathbb{Z}^\omega}(\phi(b_m, t_m, c_m))$, where $\mathrm{bij}^{-1}(b_m, t_m, c_m) \in C$. Then, $r_m \in \mathrm{proj}_{\mathbb{Z}^\omega}(\phi(b_m, t_m, c_m)) = \mathrm{proj}_{\mathbb{Z}^\omega}(\psi(c_m - t_m) +^p t_m)$ and by $(b_m, t_m, c_m) \in \mathrm{dom}(\phi)$ we get that $\mathrm{proj}_{\mathbb{Z}^\omega}(\psi(c_m - t_m)) \subset \prod_k [0, b_m(k)]$ and by our assumptions on D we have $2b_m(k) \leq |t_m(k)|$ for $k \geq m_0$. Then $|r_m(k)| \geq |t_m(k)| - |b_m(k)| \geq |t_m(k)|/2$ and so $2|r_m(k)| + 1 \geq \max\{|b_m(k)|, |t_m(k)|, |c_m(k)|\}$. By our assumptions on the m_0-long initial segments of the elements of $\mathrm{bij}(D)$, and the convergence of r_m we get that the sequence $(b_m, t_m, c_m)_{n \in \omega}$ must contain a convergent subsequence, and, as bij is a homeomorphism, $\mathrm{bij}^{-1}(b_m, t_m, c_m)$ contains such a subsequence as well. If $f \in \omega^\omega$ is its limit then of course $f \in C$, and the continuity of $\mathrm{proj}_{\mathbb{Z}^\omega} \circ \phi \circ \mathrm{bij} \restriction C$ yields that $r \in \mathrm{proj}_{\mathbb{Z}^\omega}(\phi(\mathrm{bij}(f))) \subset C'$ holds.

Second, assume that C' is Haar null. By Lemma 1.9 there exists a $b \in \omega^\omega$ such that for each $b' \geq^* b$ the measure $\mu_{b'}$ witnesses $C' \in \mathcal{HN}$. Since the set C is dominating, there exists an $f \in C$ such that $f(3m) \geq b(m)$ holds for every large enough m. Then if $\mathrm{bij}(f) = (b', t', c')$, by definition $b'(m) = f(3m)$, so $\mu_{b'}$ must witness that C' is Haar null. On the other hand, $\mathrm{proj}_{\mathbb{Z}^\omega}(\phi(b', t', c')) = \mathrm{proj}_{\mathbb{Z}^\omega}(\psi(c' - t') +^p t') = \mathrm{proj}_{\mathbb{Z}^\omega}(\psi(c' - t')) + t' \subset C'$, so $\mu_{b'}$ must witness that the set $\mathrm{proj}_{\mathbb{Z}^\omega}(\psi(c' - t'))$ is Haar null as well. But, $(b', t', c') \in \mathrm{dom}(\phi)$ holds, so by (4) we have $\mu_{b'}(\mathrm{proj}_{\mathbb{Z}^\omega}(\psi(c' - t'))) > 0$, a contradiction. $\square$

3. Upper estimate of the complexity of the Haar null ideal

As a warm up, we calculate the complexity of the codes of Haar null analytic subsets of and the complexity of the closed Haar null subsets of $\mathbb{Z}^\omega$ in the Effros Borel space.

We will also make use of another consequence of the results in [3].

Fact 3.1. *Suppose that X is a Polish space, let $A \subset \omega^\omega \times X$ be a $\mathbf{\Sigma^1_1}$ set. Then the set $S_{\mathcal{ND}} = \{x : A^x x \in \mathcal{ND}\}$ is $\mathbf{\Sigma^1_1}$-inductive.*

Proof. The fact is a consequence of the following. □

Theorem 3.2. *If $F \subset (\omega^\omega \times \omega^\omega) \times X$ is a closed set, where X is a Polish space, then the set $\{x \in X : \mathrm{proj}_1(F^x) \in \mathcal{ND}\}$ is $\mathbf{\Sigma^1_1}$-inductive.*

Proof. Let P, $f : P \to \omega^\omega \times \omega^\omega$ be as in Fact 1.26, let $H = \{(y,x) \in P \times X : (f(y), x) \in F\}$. Now H is clearly G_δ as P is G_δ and f is continuous. We note that for each $x \in X$ $\mathrm{proj}_1(F^x)$ is dominating iff I has a winning strategy in the BHS game played on F^x iff I has a winning strategy in the game $G(H^x)$. So we obtained that

$$\{x \in X : \mathrm{proj}_1(F^x) \text{ is dominating}\} = X \setminus \{x \in X : \mathrm{proj}_1(F^x) \in \mathcal{ND}\}$$

is ∂G_δ. We just need to invoke Fact 1.19, which implies that $\{x \in X : \mathrm{proj}_1(F^x) \in \mathcal{ND}\} \in \partial F_\sigma$, we are done.

□

It has been shown by Solecki [17], see also [12, 19], that the codes for the closed Haar null subsets, as well as the set $\{C \in \mathcal{F}(\mathbb{Z}^\omega) : C \in \mathcal{HN}\}$ are neither analytic nor co-analytic. (In fact, $\mathbb{Z}^\omega$ can be replaced by any non-locally compact Polish group admitting a two-sided invariant metric).

So the next result is sharp.

Theorem 3.3. *If $U \subset \mathbb{Z}^\omega \times \omega^\omega$ is a $\mathbf{\Sigma^1_1}$ set then the set $S = \{x \in \omega^\omega : U^x \in \mathcal{HN}\}$ is $\mathbf{\Sigma^1_1}$-inductive, in particular, it is $\mathbf{\Delta^1_2}$.*

Proof. By Fact 3.1 it suffices to define a $\mathbf{\Sigma^1_1}$ subset A of $\omega^\omega \times \omega^\omega$ with the property that $x \in S \iff A^x \in \mathcal{ND}$. Let

$$(h, x) \in A \iff \exists g \in \mathbb{Z}^\omega (\mu_h(U^x + g) > 0).$$

It follows from [11, Theorem 29.27] and Fact 1.7 that the set A is $\mathbf{\Sigma^1_1}$.

Let $x \in \omega^\omega$ be arbitrary and assume that $U^x \notin \mathcal{HN}$. We show that in this case $A^x = \omega^\omega$. Indeed, for any $h \in \omega^\omega$ the condition $U^x \notin \mathcal{HN}$ implies the existence of a $g \in \mathbb{Z}^\omega$ with $\mu_h(U^x + g) > 0$.

Now assume that $U^x \in \mathcal{HN}$, and towards a contradiction suppose that $A^x \notin \mathcal{ND}$. Then, we apply Lemma 1.9 to U^x and get a $b \in \omega^\omega$. Pick an $h \in A^x$ such that $h \geq^* b$. Then on the one hand μ_h should witness that U^x is Haar null, on the other hand $\mu_h(U^x + g) > 0$ for some $g \in \mathbb{Z}^\omega$, a contradiction.

□

The next theorem is an extension of a result of Saint-Raymond.

Theorem 3.4. *If $A \in \mathbf{\Sigma^1_1}(\omega^\omega \times X)$ where X is a Polish space, then $\{x \in X : A^x$ is not cofinal wrt. $\leq\}$ is $\mathbf{\Sigma^1_1}$-inductive.*

Proof. We are going to define an analytic set $B \subseteq \omega^\omega \times X$ such that for each x if A^x is cofinal wrt. $\leq$ then so is B^x, and if A^x is not cofinal, then $B^x \in \mathcal{ND}$.

Let $B' \subseteq (\omega^\omega)^\omega \times X$ be defined as $((y_i)_{i\in\omega}, x) \in B' \iff (\forall i)\ (y_i, x) \in A$, which is clearly analytic. Now fix a bijection $f = (f_0, f_1) : \omega \to \omega \times \omega$, and let $g : (\omega^\omega)^\omega \times X \to \omega^\omega \times X$ be the bijection defined as $g((y_i)_{i\in\omega}, x) = (z, x)$, where $z(j) = y_{f_0(j)}(f_1(j))$. It is straightforward to check that setting B to be the image of B' under the homeomorphism g has the required properties. □

The following are immediate now.

Corollary 3.5. *The following sets are $\mathbf{\Sigma^1_1}$-inductive.*

(1) $\{C \in \mathcal{F}(\mathbb{Z}^\omega) : C \in \mathcal{HN}\}$,
(2) $\{T \in \mathcal{P}r\mathcal{T}r(\mathbb{Z}) : [T] \in \mathcal{HN}\}$,
(3) $\{C \in \mathcal{F}(\mathbb{Z}^\omega \times \omega^\omega) : \mathrm{proj}_1 C \in \mathcal{HN}\}$,
(4) $\{T \in \mathcal{P}r\mathcal{T}r(\mathbb{Z} \times \omega) : \mathrm{proj}_1[T] \in \mathcal{HN}\}$,
(5) *the set* $\{x \in X : U^x \in \mathcal{HN}\}$, *where* $U \in \mathbf{\Sigma^1_1}(\mathbb{Z}^\omega \times X)$, X *is Polish.*

4. Lower estimate of the complexity of the Haar null ideal

The following theorem summarizes the lower bounds of the complexity of the Haar null ideal in various settings. As mentioned above, the first clause is by Matheron-Zelený [12]. The second clause improves this to completeness instead of Borel-completeness in the natural setting of trees, where this makes sense, since a Polish topology is also present. The last three clauses extend this to analytic sets instead of closed ones: we show that if we naturally parametrize the set of analytic sets, then the set of parameters for which the analytic set is Haar null is hard in the appropriate sense.

Theorem 4.1.

(1) $\{C \in \mathcal{F}(\mathbb{Z}^\omega) : C \in \mathcal{HN}\}$ *is* $\mathbf{\Sigma^1_1}$*-inductive Borel-hard,*
(2) $\{T \in \mathcal{P}r\mathcal{T}r(\mathbb{Z}) : [T] \in \mathcal{HN}\}$ *is* $\mathbf{\Sigma^1_1}$*-inductive hard,*
(3) $\{C \in \mathcal{F}(\mathbb{Z}^\omega \times \omega^\omega) : \mathrm{proj}_1 C \in \mathcal{HN}\}$ *is* $\mathbf{\Sigma^1_1}$*-inductive Borel-hard,*
(4) $\{T \in \mathcal{P}r\mathcal{T}r(\mathbb{Z} \times \omega) : \mathrm{proj}_1[T] \in \mathcal{HN}\}$ *is* $\mathbf{\Sigma^1_1}$*-inductive hard,*
(5) *There exists a universal* $\mathbf{\Sigma^1_1}$ *set* $U \subset \mathbb{Z}^\omega \times \omega^\omega$ *such that* $\{y \in \omega^\omega : U^y \in \mathcal{HN}\}$ *is* $\mathbf{\Sigma^1_1}$*-inductive hard.*

Before proving this theorem, we need a couple of lemmas. Similarly to [12] the main point will be to apply results of Saint-Raymond [16] and Solecki [17] to obtain the second clause, and the remaining four clauses will be easy corollaries.

Lemma 4.2. $\{T \in \mathcal{P}r\mathcal{T}r(\omega) : [T]$ *is not cofinal wrt.* $\leq\}$ *is* $\mathbf{\Sigma^1_1}$*-inductive hard.*

Proof. Denote $A = \{T \in \mathcal{P}r\mathcal{T}r(\omega) : [T]$ is not cofinal wrt. $\leq\}$ and also $B = \{T \in \mathcal{T}r(\omega) : [T]$ is not cofinal wrt. $\leq\}$. By [16, Theorem 6.] the set B is $\mathbf{\Sigma^1_1}$-inductive hard. Therefore it suffices to show that B is a continuous preimage of A. Let us define a map $\varphi : \mathcal{T}r \to \mathcal{P}r\mathcal{T}r$ that "concatenates finite sequences of 0s after all elements of a tree" as follows:

$$\varphi(T) = \{t^\frown 0^k : t \in T, k \in \omega\}.$$

Clearly, this is a continuous map sending trees to pruned trees, and it is also easy to check $[T]$ is not cofinal wrt. $\leq$ iff $[\varphi(T)]$ is not cofinal wrt. $\leq$, showing $B = \varphi^{-1}(A)$. □

Lemma 4.3. *$\{T \in \mathcal{P}r\mathcal{T}r(\omega) : [T]$ is not cofinal wrt. $\leq^*\} = \{T \in \mathcal{P}r\mathcal{T}r(\omega) : [T] \in \mathcal{ND}\}$ is $\mathbf{\Sigma^1_1}$-inductive hard.*

Proof. Let $C = \{T \in \mathcal{P}r\mathcal{T}r(\omega) : [T]$ is not cofinal wrt. $\leq^*\}$. We will show that A from the previous lemma is a continuous preimage of C.

Let $\omega = \cup_{i\in\omega} H_i$ be a partition of the naturals into infinitely many infinite sets, end for each i let $e_i : \omega \to H_i$ be a strictly increasing enumeration. Note that if $s \in \omega^{<\omega}$ then $\mathrm{dom}(s) \cap H_i$ is an initial segment of H_i for every i, hence $e_i^{-1}(\mathrm{dom}(s) \cap H_i)$ is an initial segment of ω for every i. Therefore, if we adopt the convention that a composition $s \circ e_i$ is defined on its largest possible domain, namely on $e_i^{-1}(\mathrm{dom}(s) \cap H_i)$, then we get that $s \circ e_i \in \omega^{<\omega}$.

Let us define a map $\varphi : \mathcal{P}r\mathcal{T}r \to \mathcal{P}r\mathcal{T}r$ that "repeats a tree infinitely many times" as follows:

$$\varphi(T) = \{s \in \omega^{<\omega} : \forall i \ s \circ e_i \in T\}.$$

Clearly, this is a continuous map sending pruned trees to pruned trees, and it is also not hard to check that $[T]$ is not cofinal wrt. $\leq$ iff $[\varphi(T)]$ is not cofinal wrt. $\leq^*$, showing $A = \varphi^{-1}(C)$. □

The next preparation we need before proving Theorem 4.1 is a version of a result of Solecki [17, Theorem 2.1.]. On the one hand, we only need the special case when the group is $\mathbb{Z}^\omega$, but we also need some extra information that can actually be extracted from Solecki's proof, however, that proof is rather hard and technical, so we have decided to include a simpler form here

Lemma 4.4. *There exist disjoint intervals $I_0, I_1, \ldots \subset \mathbb{Z}$ such that $|I_k| = k+1$ for every k, and for every interval $J \subset \mathbb{Z}$*

$$|\{k \in \omega : J \cap I_k \neq \emptyset\}| \geq 2 \implies \frac{|J \cap \cup_k I_k|}{|J|} \leq \frac{1}{2}.$$

Proof. It is easy to see that $I_k = [a_k, a_k + k]$ works if a_k is a rapidly increasing sequence. □

Theorem 4.5. *There exists a closed set $F \subset \mathbb{Z}^\omega$ and a continuous open surjective function $S : F \to \omega^\omega$ such that for $A \subset \omega^\omega$*

$$A \in \mathcal{ND} \iff S^{-1}(A) \in \mathcal{HN}.$$

Moreover, if $[s]$ is a basic open set in $\mathbb{Z}^\omega$ then $S([s] \cap F)$ is basic open.

Proof. Let I_k be the intervals from the previous lemma, and let us define

$$F = (\cup_k I_k)^\omega.$$

Then F is clearly closed.

Now, for every $x \in F$ there exists a unique $y \in \omega^\omega$ such that $x(n) \in I_{y(n)}$ for every n, and let us define

$$S(x) = y.$$

Clearly, S is continuous and surjective. It is also clear from the definition that the image of a basic open set is basic open, which in turn implies that f is open. Hence it remains to check that $A \in \mathcal{ND} \iff S^{-1}(A) \in \mathcal{HN}$.

First we check that if A is dominating then $S^{-1}(A)$ is compact-biter, hence not Haar null. It clearly suffices to check this for compact sets of the form

$$\times_n [0, y(n)],$$

where $y \in (\omega \setminus \{0\})^\omega$. Since A is dominating, there exists $x \in A$ and $N \in \omega$ such that $x(n) \geq y(n)$ for $n \geq N$. But then $|I_{x(n)}| \geq |[0, y(n)]|$ for $n \geq N$, hence it is easy to construct a translation $t \in \mathbb{Z}^\omega$ such that

$$\{z \in \times_n [0, y(n)] : z(n) = 0 \text{ for every } n < N\} + t \subset \times_n I_{x(n)} =$$

$$= S^{-1}(x) \subset S^{-1}(A),$$

hence a (relative) basic open subset of $\times_n [0, y(n)]$ can be translated into $S^{-1}(A)$.

Next we check that if A is not dominating then $S^{-1}(A)$ is Haar null. Let $y \in \omega^\omega$ such that

$$\forall x \in A\ \exists^\infty n \text{ with } y(n) > x(n). \tag{4.1}$$

By replacing A with the larger not dominating G_δ set

$$\{x \in \omega^\omega :\ \exists^\infty n \text{ with } y(n) > x(n)\}$$

we may assume that A is G_δ, hence $S^{-1}(A)$ is also G_δ, therefore universally measurable.

We claim that the measure μ_{2y} will witness that $S^{-1}(A)$ is Haar null. Let $t \in \mathbb{Z}^\omega$ be a translation. We need to show that

$$\mu_{2y}(S^{-1}(A) + t) = 0. \tag{4.2}$$

Case 1. $\exists^\infty n$ such that $|\{k \in \omega : [0, 2y(n)] \cap (I_k + t(n)) \neq \emptyset\}| \geq 2$.

Then, for every such n applying the previous lemma with $J = [-t(n), 2y(n) - t(n)]$ yields that

$$\frac{|[0, 2y(n)] \cap ((\cup_k I_k) + t(n))|}{2y(n) + 1} \leq \frac{1}{2},$$

hence the "measure is halved" at infinitely many coordinates, therefore even

$$\mu_{2y}((\cup_k I_k)^\omega + t) = 0$$

holds, hence $(\cup_k I_k)^\omega = F \supset S^{-1}(A)$ proves (4.2).

Case 2. $\forall^\infty n$ we have $|\{k \in \omega : [0, 2y(n)] \cap (I_k + t(n)) \neq \emptyset\}| \leq 1$.

Then there are only countably many $x \in \omega^\omega$ such that

$$(\times_n [0, 2y(n)]) \cap (\times_n (I_{x(n)} + t(n)) \neq \emptyset.$$

Therefore, since

$$S^{-1}(A) = \cup_{x \in A} \times_n (I_{x(n)} + t(n)),$$

it suffices to prove that for every $x \in A$ we have

$$\mu_{2y}(\times_n (I_{x(n)} + t(n))) = 0.$$

But this is clear, as by (4.1) the measure is again "halved infinitely many times". □

Lemma 4.6. *There exists a continuous map $\varphi : \mathcal{P}r\mathcal{T}r(\omega) \to \mathcal{P}r\mathcal{T}r(\mathbb{Z})$ such that*

$$[T] \in \mathcal{ND} \iff [\varphi(T)] \in \mathcal{HN}.$$

Proof. Let $\varphi(T)$ be the unique pruned tree such that

$$[\varphi(T)] = S^{-1}([T]),$$

that is, for every $s \in \mathbb{Z}^{<\omega}$ we have $s \in \varphi(T)$ iff $[s] \cap S^{-1}([T]) \neq \emptyset$.

Then $[T] \in \mathcal{ND} \iff [\varphi(T)] \in \mathcal{HN}$ is immediate from the previous theorem.

Hence, the last thing to be proved is continuity. We have to show that for every $s \in \mathcal{P}r\mathcal{T}r(\mathbb{Z})$ the sets $\varphi^{-1}(\{T' \in \mathcal{P}r\mathcal{T}r(\mathbb{Z}) : s \in T'\})$ and $\varphi^{-1}(\{T' \in \mathcal{P}r\mathcal{T}r(\mathbb{Z}) : s \notin T'\})$ are open in $\mathcal{P}r\mathcal{T}r(\omega)$.

$$\varphi^{-1}(\{T' \in \mathcal{P}r\mathcal{T}r(\mathbb{Z}) : s \in T'\}) = \{T \in \mathcal{P}r\mathcal{T}r(\omega) : s \in \varphi(T)\} =$$
$$\{T \in \mathcal{P}r\mathcal{T}r(\omega) : [s] \cap [\varphi(T)] \neq \emptyset\} =$$
$$\{T \in \mathcal{P}r\mathcal{T}r(\omega) : [s] \cap S^{-1}([T]) \neq \emptyset\} =$$
$$\{T \in \mathcal{P}r\mathcal{T}r(\omega) : S([s]) \cap [T] \neq \emptyset\},$$

which is easily seen to be open, since $S([s])$ is open.

Similarly,

$$\varphi^{-1}(\{T' \in \mathcal{P}r\mathcal{T}r(\mathbb{Z}) : s \notin T'\}) =$$
$$\{T \in \mathcal{P}r\mathcal{T}r(\omega) : S([s]) \cap [T] = \emptyset\},$$

which is also open using that by the last clause of the previous theorem $S([s]) = [t]$ for some $t \in \omega^{<\omega}$. □

Now we are ready to prove the main theorem of the section.

Proof. (Theorem 4.1)

(2) By the previous lemma we obtain that the continuous preimage

$$\varphi^{-1}(\{T \in \mathcal{P}r\mathcal{T}r(\mathbb{Z}) : [T] \in \mathcal{HN}\}) = \{T \in \mathcal{P}r\mathcal{T}r(\omega) : [T] \in \mathcal{ND}\},$$

which is a $\mathbf{\Sigma^1_1}$-inductive hard set by Lemma 4.3, hence $\{T \in \mathcal{P}r\mathcal{T}r(\mathbb{Z}) : [T] \in \mathcal{HN}\}$ is also $\mathbf{\Sigma^1_1}$-inductive hard.

(1) Here we only use the well-known and easy fact that the map $T \mapsto [T]$ is a Borel-isomorphism between $\mathcal{P}r\mathcal{T}r(\mathbb{Z})$ and $\mathcal{F}(\mathbb{Z}^\omega)$, hence it preserves $\mathbf{\Sigma^1_1}$-inductive Borel-hardness by Lemma 1.18.

(3) The map $\psi : \mathcal{F}(\mathbb{Z}^\omega) \to \mathcal{F}(\mathbb{Z}^\omega \times \omega^\omega)$ defined as

$$\psi(C) = C \times \omega^\omega$$

is easily seen to be Borel, and clearly

$$\psi^{-1}(\{C \in \mathcal{F}(\mathbb{Z}^\omega \times \omega^\omega) : \mathrm{proj}_1 C \in \mathcal{HN}\}) = \{C \in \mathcal{F}(\mathbb{Z}^\omega) : C \in \mathcal{HN}\},$$

hence $\{C \in \mathcal{F}(\mathbb{Z}^\omega \times \omega^\omega) : \mathrm{proj}_1 C \in \mathcal{HN}\}$ is $\mathbf{\Sigma^1_1}$-inductive Borel-hard by (1).

(4) Analogously, the map $\psi' : \mathcal{P}r\mathcal{T}r(\mathbb{Z}) \to \mathcal{P}r\mathcal{T}r(\mathbb{Z} \times \omega)$ defined as

$$\psi'(T) = \cup_{t \in T}(\{t\} \times \omega^{|t|})$$

is easily seen to be continuous, and clearly

$$\mathrm{proj}_1[\psi'(T)] = [T] \text{ for every } T,$$

hence

$$\psi'^{-1}(\{T \in \mathcal{P}r\mathcal{T}r(\mathbb{Z} \times \omega) : \mathrm{proj}_1[T] \in \mathcal{HN}\}) =$$
$$\{T \in \mathcal{P}r\mathcal{T}r(\mathbb{Z}) : [T] \in \mathcal{HN}\},$$

hence $\{T \in \mathcal{P}r\mathcal{T}r(\mathbb{Z} \times \omega) : \mathrm{proj}_1[T] \in \mathcal{HN}\}$ is $\mathbf{\Sigma^1_1}$-inductive hard by (2).

(5) By Lemma 1.4 ω^ω is homeomorphic to $\mathcal{P}r\mathcal{T}r(\mathbb{Z} \times \omega)$, hence it suffices to prove that there exists a universal $\mathbf{\Sigma^1_1}$ set $U \subset \mathbb{Z}^\omega \times \mathcal{P}r\mathcal{T}r(\mathbb{Z} \times \omega)$ such that $\{T \in \mathcal{P}r\mathcal{T}r(\mathbb{Z} \times \omega) : U^T \in \mathcal{HN}\}$ is $\mathbf{\Sigma^1_1}$-inductive hard.

Define

$$U = \{(x, T) \in \mathbb{Z}^\omega \times \mathcal{P}r\mathcal{T}r(\mathbb{Z} \times \omega) : x \in \mathrm{proj}_1[T]\}.$$

Then it is easy to see that U is $\mathbf{\Sigma^1_1}$, and the horizontal section $U^T = \mathrm{proj}_1[T]$ for every T. Hence, on the one hand, U is universal, and on the other hand,

$U^T \in \mathcal{HN} \iff \mathrm{proj}_1[T] \in \mathcal{HN}$, therefore $\{T \in \mathcal{P}r\mathcal{T}r(\mathbb{Z} \times \omega) : U^T \in \mathcal{HN}\}$ is $\mathbf{\Sigma^1_1}$-inductive hard by (4). $\square$

As a spin-off, combining Theorem 3.4 with the next theorem we obtain a more general version of the above mentioned result of Saint-Raymond: one can parametrize the class of $\mathbf{\Sigma^1_1}$ subsets of ω^ω (using an universal $\mathbf{\Sigma^1_1}$ set) such that the parameters corresponding to the sets that are not cofinal wrt. $\leq$ form a $\mathbf{\Sigma^1_1}$-inductive complete set.

Theorem 4.7. *There exists a universal $\mathbf{\Sigma^1_1}$ set $U \subset \omega^\omega \times \omega^\omega$ such that $\{y \in \omega^\omega : U^y$ is not cofinal wrt. $\leq\}$ is $\mathbf{\Sigma^1_1}$-inductive hard.*

Proof. The proof is essentially the same as that of (4) and (5) above.

By Lemma 4.2,

$$\{T \in \mathcal{P}r\mathcal{T}r(\mathbb{Z}) : [T] \text{ is not cofinal wrt. } \leq\}$$

is $\mathbf{\Sigma^1_1}$-inductive hard.

Let $\psi' : \mathcal{P}r\mathcal{T}r(\omega) \to \mathcal{P}r\mathcal{T}r(\omega \times \omega)$ defined as

$$\psi'(T) = \cup_{t \in T}(\{t\} \times \omega^{|t|}).$$

Again

$$\psi'^{-1}(\{T \in \mathcal{P}r\mathcal{T}r(\omega \times \omega) : \mathrm{proj}_1[T] \text{ is not cofinal wrt. } \leq\}) = \{T \in \mathcal{P}r\mathcal{T}r(\omega) : [T] \text{ is not cofinal wrt. } \leq\},$$

hence

$$\{T \in \mathcal{P}r\mathcal{T}r(\omega \times \omega) : \mathrm{proj}_1[T] \text{ is not cofinal wrt. } \leq\} \text{ is } \mathbf{\Sigma^1_1}\text{-inductive hard.} \tag{4.3}$$

As above, by Lemma 1.4 it suffices to prove that there exists a universal $\mathbf{\Sigma^1_1}$ set $U \subset \omega^\omega \times \mathcal{P}r\mathcal{T}r(\omega \times \omega)$ such that $\{T \in \mathcal{P}r\mathcal{T}r(\omega \times \omega) : U^T$ is not cofinal wrt. $\leq\}$ is $\mathbf{\Sigma^1_1}$-inductive hard.

Define

$$U = \{(x, T) \in \omega^\omega \times \mathcal{P}r\mathcal{T}r(\omega \times \omega) : x \in \mathrm{proj}_1[T]\}.$$

Then again, $U^T = \mathrm{proj}_1[T]$ for every T, hence,

$$U^T \text{ is not cofinal wrt. } \leq \iff \mathrm{proj}_1[T] \text{ is not cofinal wrt. } \leq,$$

therefore $\{T \in \mathcal{P}r\mathcal{T}r(\omega \times \omega) : U^T$ is not cofinal wrt. $\leq\}$ is $\mathbf{\Sigma^1_1}$-inductive hard by (4.3). $\square$

5. Open questions

Let us finish the paper with a couple of open problems.

Problem 5.1. *In Theorem 4.1 (5) we have shown that there exist a universal $\mathbf{\Sigma^1_1}$ set $U \subset \mathbb{Z}^\omega \times \omega^\omega$ such that $\{y \in \omega^\omega : U^y \in \mathcal{HN}\}$ is $\mathbf{\Sigma^1_1}$-inductive hard. Is this true for* every *universal $\mathbf{\Sigma^1_1}$ set?*

The following problem has already been asked several times, here we repeat it since we believe it may be closely related to our result about closed regularity of the Haar null ideal.

Problem 5.2. *Let $\mathbb{P}$ be the poset of non-Haar null Borel subsets of $\mathbb{Z}^\omega$ ordered under inclusion. Is this a proper forcing notion?*

And finally,

Problem 5.3. *Does Theorem 0.2 remain true if we replace $\mathbb{Z}^\omega$ by other Polish groups? What about the case when G admits a two-sided invariant metric? Or when G is a separable Banach space?*

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